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Dubrovin's duality for F-manifolds with eventual identities

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Abstract

A vector field $\mathcal{E}$ on an F-manifold $(M, \circ, e)$ is an eventual identity if it is invertible and the multiplication $X * Y := X \circ Y \circ \mathcal{E}^{-1}$ defines a new F-manifold structure on M. We give a characterization of such eventual identities, this being a problem raised by Manin (2005) [12]. We develop a duality between F-manifolds with eventual identities and we show that this duality is compatible with the local irreducible decomposition of F-manifolds and preserves the class of Riemannian F-manifolds. We find necessary and sufficient conditions on the eventual identity which ensure that the classes of harmonic Higgs bundles, $DChk$-structures and weak CV-structures are preserved by our duality. Examples of such structures are given in the case of a semi-simple multiplication. We use eventual identities to construct compatible pairs of metrics.

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Contents

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1. Introduction

In [3] Dubrovin introduced the idea of an almost dual Frobenius manifold. Starting from a Frobenius manifold one may construct a new geometric object that shares many, but crucially not all, of the essential features of the original manifold. In particular a new 'dual' solution of the underlying Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations may be constructed from the original manifold. Such a construction reflects certain other 'dualities' that occur in other areas of mathematics where Frobenius manifolds appear. For example, in:

- Quantum cohomology and mirror symmetry;
- Integrable systems, via generalizations of the classical Miura transform;
- Singularity theory, via the correspondence between oscillatory integrals and period integrals.

More specifically, given a Frobenius manifold $(M, \circ, e, E, \tilde{g})$ with multiplication $\circ$, unit field e, Euler field E and metric $\tilde{g}$ one may define a new multiplication $*$ and metric g by the formulae

$$X * Y = X \circ Y \circ E^{-1},$$
$$g(X, Y) = \tilde{g}\big(E^{-1} \circ X, Y\big)$$

where $E^{-1} \circ E = e$. Clearly $*$ is associative, commutative and has a unit field, namely E, the original Euler field. The new metric g (the intersection form) turns out to be flat and from these two new objects one may define a dual solution to the WDVV equations. This correspondence is not completely dual – certain properties are lost. For example, while $\tilde{\nabla} e = 0$, the new identity does not share this property: in general $\nabla E \neq 0$.

Underlying Frobenius manifolds is a structure known as an F-manifold, which was introduced by Hertling and Manin [8].

Definition 1. (See [8].)

i) An F-manifold is a triple $(M, \circ, e)$ where M is a manifold, $\circ$ is a $C^\infty(M)$-bilinear, commutative, associative multiplication on the tangent bundle TM, with unit field e, such that the F-manifold condition

$$L_{X\circ Y}(\circ) := X \circ L_Y(\circ) + Y \circ L_X(\circ), \tag{1}$$

holds, for any smooth vector fields $X, Y \in \mathcal{X}(M)$.

ii) An Euler vector field on an F-manifold $(M, \circ, e)$ is a vector field E which preserves the multiplication up to a constant, i.e.

$$L_E(\circ)(X, Y) = dX \circ Y, \quad \forall X, Y \in \mathcal{X}(M).$$

The constant d is called the weight of E.

F-manifolds appear in many areas of mathematics. All Frobenius manifolds have an underlying F-manifold structure, and in examples originating from singularity theory such F-manifolds arise in a very natural way [7]. They also appear within integrable systems – both in examples coming from the submanifold geometry of Frobenius manifolds [17] and non-local bi-Hamiltonian geometry [2] and their role has been elucidated further in [10].

Given an F-manifold with an invertible Euler vector field one may construct a dual multiplication via $X * Y = X \circ Y \circ E^{-1}$. While this is bilinear, commutative and associative with unit field, whether or not this defines an F-manifold is not immediately clear. More generally, Manin [12] replaced the Euler field E by an arbitrary invertible vector field and used this to define a new multiplication.

Definition 2. (See [12].) A vector field $\mathcal{E}$ on an F-manifold $(M, \circ, e)$ is called an eventual identity if it is invertible (i.e. there is a vector field $\mathcal{E}^{-1}$ such that $\mathcal{E} \circ \mathcal{E}^{-1} = \mathcal{E}^{-1} \circ \mathcal{E} = e$) and, moreover, the multiplication

$$X * Y = X \circ Y \circ \mathcal{E}^{-1}, \quad \forall X, Y \in \mathcal{X}(M) \tag{2}$$

defines a new F-manifold structure on M.

The reason for the terminology is that $\mathcal{E}$ is the unit field for the multiplication $*$. In this paper we give the characterization of such eventual identities, thus answering a question raised by Manin [12].

Theorem 3.

i) *Let $(M, \circ, e)$ be an F-manifold and $\mathcal{E}$ an invertible vector field. Then $\mathcal{E}$ is an eventual identity if and only if*

$$L_{\mathcal{E}}(\circ)(X, Y) = [e, \mathcal{E}] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M). \tag{3}$$

ii) *Assume that* (3) *holds and let*

$$X * Y = X \circ Y \circ \mathcal{E}^{-1}$$

be the new F-manifold multiplication. Then e is an eventual identity on $(M, *, \mathcal{E})$ and the map

$$(M, \circ, e, \mathcal{E}) \to (M, *, \mathcal{E}, e)$$

is an involution on the set of F-manifolds with eventual identities.

Condition (3) above may be seen as a generalization of the notion of an Euler vector field. All invertible Euler vector fields are eventual identities but not conversely. However, eventual identities play a similar role. In this paper we study F-manifolds with eventual identities and their relation with some well-known constructions in the theory of Frobenius manifolds.

The plan of the paper is the following. In Section 2 we prove Theorem 3 and we develop its consequences. We remark that the duality for F-manifolds with eventual identities developed in Theorem 3 ii) is a natural generalization of the well-known dualities for almost Frobenius manifolds and for F-manifolds with compatible flat structures [3,12]. After proving Theorem 3 we show that any eventual identity on a product F-manifold is a sum of eventual identities on the factors (a similar decomposition holds for Euler vector fields [7]). Using this fact we show that our duality for F-manifolds with eventual identities is compatible with the local irreducible decomposition of F-manifolds developed in [7]. We end Section 2 with examples and further properties of eventual identities, some of them being already known for Euler vector fields.

In Section 3 we add a new ingredient on our F-manifold $(M, \circ, e, \mathcal{E})$ with eventual identity, namely a multiplication invariant metric $\tilde{g}$. The eventual identity $\mathcal{E}$ together with $\tilde{g}$ determine, in a canonical way, a second metric g, defined like the second metric of a Frobenius manifold. We prove that the metrics $(g, \tilde{g})$ are almost compatible (see Proposition 10). Our main result in this section states that $(g, \tilde{g})$ are compatible, when $(M, \circ, e, \tilde{g})$ is an almost Riemannian F-manifold, i.e. the coidentity $\epsilon \in \Omega^1(M)$, which is the 1-form dual to the unit field e, is closed (see Theorem 12). Similar results already appear in the literature [2], with Euler vector fields instead of eventual identities.

In Section 4 we show that our duality for F-manifolds with eventual identities preserves the class of Riemannian F-manifolds, which are almost Riemannian F-manifolds satisfying an additional curvature condition (see Definition 15 and Theorem 16). Riemannian F-manifolds were introduced and studied in [10] and are closely related to the theory of integrable systems of hydrodynamic type.

In Section 5 we apply our results from Sections 3 and 4 to the theory of integrable systems. In particular it is shown how the principal hierarchy of dispersionless integrable systems is preserved under this duality – this generalizes the Miura-type transformation behind Dubrovin's original construction [3]. In the semi-simple case (equivalent to the existence of Riemann invariants for the system) the theory reduces to the study of Tsarev's equation [19], but in addition it explains the geometrical origins of the functional freedom in the solution of Tsarev's equation in terms of eventual identities and the preservation of the underlying F-manifold structures.

In Section 6 we study the interactions between tt^*-geometry and our duality for F-manifolds with eventual identities. The main notion in tt^*-geometry is the so called CV-structure [1], which shares many properties in common with the notion of Frobenius structure, its main ingredients being a metric, a Higgs field and a real structure (the latter not being present in the theory of Frobenius manifolds). One often considers weaker structures, like harmonic Higgs bundles [16] or $DChk$-structures [6] (i.e. harmonic Higgs bundles with compatible real structures). One can combine tt^*-geometry with Frobenius manifold theory giving rise to new structures (like CDV-structures) satisfying some complicated compatibility conditions, but which are very natural in examples coming from singularity theory. It is in this context that F-manifolds appear in tt^*-geometry. In the same framework like in Sections 3 and 4, we add structures – Hermitian metrics and real structures – on an holomorphic F-manifold $(M, \circ, e)$ and we study their behavior under twisting by an eventual identity. We assume that these structures are compatible with the multiplication $\circ$, i.e. they form harmonic Higgs bundles or $DChk$-structures, and we determine necessary and sufficient conditions on the eventual identity such that the resulting dual structures are compatible in the same way (see conditions (60) and (62) of Theorem 22). The class of CV-structures turns out to be rigid – it is never preserved by the duality from Theorem 3. With this motivation we define a weaker notion of CV-structure, essentially by replacing the Euler vector field associated to a CV-structure by an eventual identity (see Definition 25 and Lemma 26). We prove that the more general class of weak CV-structures is preserved by the duality for F-manifolds with eventual identities, provided that the same conditions (60) and (62) on the eventual identity are satisfied (see Theorem 28). At the end of this section we consider the simplest case – when the F-manifold is semi-simple and the metric and real structure are diagonal – and we show that the conditions (60) and (62) are automatically satisfied. Thus all classes – harmonic Higgs bundles, $DChk$-structures and weak CV-structures – are preserved by the duality in this case.

2. Eventual identities and duality

In this section we prove Theorem 3. We begin with a simple preliminary lemma concerning invertible vector fields on F-manifolds.

Lemma 4. *Let $(M, \circ, e)$ be an F-manifold and $\mathcal{E}$ an invertible vector field, with inverse $\mathcal{E}^{-1}$. Assume that*

$$L_{\mathcal{E}}(\circ)(X, Y) = [e, \mathcal{E}] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M). \tag{4}$$

Then also

$$L_{\mathcal{E}^{-1}}(\circ)(X, Y) = \left[e, \mathcal{E}^{-1}\right] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M). \tag{5}$$

Proof. The proof is a simple calculation. Since $e = e \circ e$, the F-manifold condition (1) with $X = Y := e$ implies that $L_e(\circ) = 0$. Applying again (1) with $X := \mathcal{E}$ and $Y := \mathcal{E}^{-1}$, we obtain:

$$0 = L_{\mathcal{E} \circ \mathcal{E}^{-1}}(\circ) = \mathcal{E} \circ L_{\mathcal{E}^{-1}}(\circ) + \mathcal{E}^{-1} \circ L_{\mathcal{E}}(\circ).$$

Combining this relation with (4) we get

$$L_{\mathcal{E}^{-1}}(\circ)(X, Y) = \mathcal{E}^{-2} \circ [\mathcal{E}, e] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M),$$

where $\mathcal{E}^{-2}$ denotes $\mathcal{E}^{-1} \circ \mathcal{E}^{-1}$. On the other hand,

$$[e, \mathcal{E}] \circ \mathcal{E}^{-2} = \left(L_e(\mathcal{E}) \circ \mathcal{E}^{-1}\right) \circ \mathcal{E}^{-1} = \left(L_e(e) - \mathcal{E} \circ L_e\left(\mathcal{E}^{-1}\right)\right) \circ \mathcal{E}^{-1} = \left[\mathcal{E}^{-1}, e\right]$$

where we used $L_e(\circ) = 0$. Our claim follows. □

Note that the construction of $\mathcal{E}^{-1}$, whilst just linear algebra, requires the inversion of a matrix, and hence $\mathcal{E}^{-1}$ is not defined at points of M where a certain determinant Σ vanishes. Rather than defining a new manifold $M^{\star} = M \backslash \Sigma$ on which $\mathcal{E}^{-1}$ is defined we just assume that M consists of points at which both $\mathcal{E}$ and $\mathcal{E}^{-1}$ are well defined.

After this preliminary result, we now prove Theorem 3 stated in the Introduction.

Proof of Theorem 3. The multiplication $*$ defined by (2) is $C^\infty(M)$-bilinear, commutative, associative, with unit field $\mathcal{E}$. Therefore $(M, *, \mathcal{E})$ is an F-manifold if and only if for any vector fields $Z, V \in \mathcal{X}(M)$,

$$L_{Z*V}(*)(X, Y) = Z * L_V(*)(X, Y) + V * L_Z(*)(X, Y), \quad \forall X, Y \in \mathcal{X}(M). \tag{6}$$

We will show that (6) is equivalent with (3). For this, we take the Lie derivative with respect to Z of the relation (2). We get, by a straightforward computation,

$$L_Z(*)(X, Y) = L_Z(\circ)\big(\mathcal{E}^{-1} \circ X, Y\big) + L_Z(\circ)\big(\mathcal{E}^{-1}, X\big) \circ Y + \big[Z, \mathcal{E}^{-1}\big] \circ X \circ Y. \tag{7}$$

Using relation (7) with Z replaced by $Z * V = Z \circ V \circ \mathcal{E}^{-1}$ and the F-manifold condition (1) satisfied by the multiplication $\circ$, we get:

$$\begin{aligned} L_{Z*V}(*)(X, Y) = {} & \mathcal{E}^{-1} \circ Z \circ L_V(\circ)\big(\mathcal{E}^{-1} \circ X, Y\big) + \mathcal{E}^{-1} \circ V \circ L_Z(\circ)\big(\mathcal{E}^{-1} \circ X, Y\big) \\ & + Z \circ V \circ L_{\mathcal{E}^{-1}}(\circ)\big(\mathcal{E}^{-1} \circ X, Y\big) + \mathcal{E}^{-1} \circ Z \circ Y \circ L_V(\circ)\big(\mathcal{E}^{-1}, X\big) \\ & + \mathcal{E}^{-1} \circ V \circ Y \circ L_Z(\circ)\big(\mathcal{E}^{-1}, X\big) + Z \circ V \circ Y \circ L_{\mathcal{E}^{-1}}(\circ)\big(\mathcal{E}^{-1}, X\big) \\ & - L_{\mathcal{E}^{-1}}\big(\mathcal{E}^{-1} \circ Z \circ V\big) \circ X \circ Y. \end{aligned}$$

Combining this expression with the expressions of $L_Z(*)(X, Y)$ and $L_V(*)(X, Y)$ provided by (7), we see that (6) holds if and only if

$$\begin{aligned} & X \circ Y \circ \big(L_{\mathcal{E}^{-1}}\big(\mathcal{E}^{-1} \circ Z \circ V\big) + \mathcal{E}^{-1} \circ Z \circ \big[V, \mathcal{E}^{-1}\big] + \mathcal{E}^{-1} \circ V \circ \big[Z, \mathcal{E}^{-1}\big]\big) \\ & \quad = Z \circ V \circ \big(L_{\mathcal{E}^{-1}}(\circ)\big(\mathcal{E}^{-1} \circ X, Y\big) + Y \circ L_{\mathcal{E}^{-1}}(\circ)\big(\mathcal{E}^{-1}, X\big)\big). \end{aligned}$$

On the other hand, it can be checked that

$$\begin{aligned} & L_{\mathcal{E}^{-1}}\big(\mathcal{E}^{-1} \circ Z \circ V\big) + \mathcal{E}^{-1} \circ Z \circ \big[V, \mathcal{E}^{-1}\big] + \mathcal{E}^{-1} \circ V \circ \big[Z, \mathcal{E}^{-1}\big] \\ & \quad = L_{\mathcal{E}^{-1}}(\circ)\big(\mathcal{E}^{-1}, Z\big) \circ V + L_{\mathcal{E}^{-1}}(\circ)\big(\mathcal{E}^{-1} \circ Z, V\big). \end{aligned}$$

Hence $*$ is the multiplication of an F-manifold structure if and only if for any vector fields $X, Y, Z, V \in \mathcal{X}(M)$,

$$\begin{aligned} & X \circ Y \circ \big(L_{\mathcal{E}^{-1}}(\circ)\big(\mathcal{E}^{-1} \circ Z, V\big) + L_{\mathcal{E}^{-1}}(\circ)\big(\mathcal{E}^{-1}, Z\big) \circ V\big) \\ & \quad = Z \circ V \circ \big(L_{\mathcal{E}^{-1}}(\circ)\big(\mathcal{E}^{-1} \circ X, Y\big) + L_{\mathcal{E}^{-1}}(\circ)\big(\mathcal{E}^{-1}, X\big) \circ Y\big). \end{aligned}$$

Taking $X = Y := e$ it is easy to see that this relation is equivalent with

$$L_{\mathcal{E}^{-1}}(\circ)\big(\mathcal{E}^{-1} \circ Z, V\big) + L_{\mathcal{E}^{-1}}(\circ)\big(\mathcal{E}^{-1}, Z\big) \circ V = -2\mathcal{E}^{-1} \circ \big[\mathcal{E}^{-1}, e\big] \circ Z \circ V. \tag{8}$$

We now simplify relation (8). For this, we take in (8) $Z := e$ and we obtain

$$L_{\mathcal{E}^{-1}}(\circ)\big(\mathcal{E}^{-1}, V\big) = -\mathcal{E}^{-1} \circ \big[\mathcal{E}^{-1}, e\big] \circ V, \quad \forall V \in \mathcal{X}(M). \tag{9}$$

Combining (8) with (9) we get:

$$L_{\mathcal{E}^{-1}}(\circ)(Z, V) = -\big[\mathcal{E}^{-1}, e\big] \circ Z \circ V, \quad \forall Z, V \in \mathcal{X}(M). \tag{10}$$

Conversely, it is clear that if (10) is satisfied then (8) is satisfied as well. Therefore, relations (8) and (10) are equivalent. We proved that $*$ is the multiplication of an F-manifold structure if and only if (10) holds. Our first claim follows from Lemma 4.

For our second claim, assume that $\mathcal{E}$ is an eventual identity on an F-manifold $(M, \circ, e)$. We want to prove that e is an eventual identity on the F-manifold $(M, *, \mathcal{E})$, where $*$ is related to $\circ$ by (2). Since the unit field of $*$ is $\mathcal{E}$, we need to show that

$$L_e(*)(X, Y) = [\mathcal{E}, e] * X * Y, \quad \forall X, Y \in \mathcal{X}(M). \tag{11}$$

Letting $Z := e$ in (7) and using $L_e(\circ) = 0$ together with (2), we get:

$$L_e(*)(X, Y) = \big[e, \mathcal{E}^{-1}\big] \circ X \circ Y = \big(\big[e, \mathcal{E}^{-1}\big] \circ \mathcal{E}^2\big) * X * Y.$$

Recall now from the proof of Lemma 4 that $[e, \mathcal{E}^{-1}] \circ \mathcal{E}^2 = [\mathcal{E}, e]$. Our second claim follows. The proof of Theorem 3 is now completed. $\square$

Having found the characterization of eventual identities one may study how such objects can be combined to form new eventual identities.

Proposition 5.

i) *Eventual identities form a subgroup of the group of invertible vector fields on an F-manifold.*
ii) *The Lie bracket of two eventual identities is an eventual identity, provided that is invertible.*
iii) *Let $(M_1 \times M_2, \circ, e_1 + e_2)$ be the product of two F-manifolds $(M_1, \circ_1, e_1)$ and $(M_2, \circ_2, e_2)$, with multiplication defined by*

$$(X_1, X_2) \circ (Y_1, Y_2) = (X_1 \circ_1 Y_1, X_2 \circ_2 Y_2), \tag{12}$$

for any $X_1, Y_1 \in \mathcal{X}(M_1)$ and $X_2, Y_2 \in \mathcal{X}(M_2)$ (considered as vector fields on $M_1 \times M_2$). If $\mathcal{E}_1$ is an eventual identity on $(M, \circ_1, e_1)$ and $\mathcal{E}_2$ is an eventual identity on $(M, \circ, e_2)$, then $\mathcal{E} := \mathcal{E}_1 + \mathcal{E}_2$ is an eventual identity on $(M_1 \times M_2, \circ, e_1 + e_2)$. Moreover, any eventual identity on $(M_1 \times M_2, \circ, e_1 + e_2)$ is obtained in this way.

Proof. i) If $\mathcal{E}_1$ and $\mathcal{E}_2$ are eventual identities then $\mathcal{E}_1 \circ \mathcal{E}_2$ is invertible and for any $X, Y \in \mathcal{X}(M)$,

$$\begin{aligned} L_{\mathcal{E}_1 \circ \mathcal{E}_2}(\circ)(X, Y) &= \mathcal{E}_1 \circ L_{\mathcal{E}_2}(\circ)(X, Y) + \mathcal{E}_2 \circ L_{\mathcal{E}_1}(\circ)(X, Y) \\ &= \big(\mathcal{E}_1 \circ [e, \mathcal{E}_2] + \mathcal{E}_2 \circ [e, \mathcal{E}_1]\big) \circ X \circ Y \\ &= [e, \mathcal{E}_1 \circ \mathcal{E}_2] \circ X \circ Y \end{aligned}$$

where in the last equality we used $L_e(\circ) = 0$. Thus $\mathcal{E}_1 \circ \mathcal{E}_2$ is an eventual identity, from Theorem 3. Moreover, from Lemma 4 and Theorem 3 again, if $\mathcal{E}$ is an eventual identity then also $\mathcal{E}^{-1}$ is an eventual identity. Our first claim follows.

ii) Recall the following relation proved in Proposition 4.3 of [8]: for any vector fields $X, Y, Z, W \in \mathcal{X}(M)$,

$$L_{[X,Y]}(\circ)(Z,W) = \big[X, L_Y(\circ)(Z,W)\big] - L_Y(\circ)\big([X,Z],W\big) - L_Y(\circ)\big(Z,[X,W]\big)$$
$$- \big[Y, L_X(\circ)(Z,W)\big] + L_X(\circ)\big([Y,Z],W\big) + L_X(\circ)\big(Z,[Y,W]\big).$$

Our second claim follows this relation and Theorem 3.

iii) It is straightforward to check that a sum of eventual identities on the factors gives an eventual identity on the product $(M_1 \times M_2, \circ, e_1 + e_2)$. The converse is more involved and goes as follows (a similar argument has been used for the decomposition of Euler vector fields on product F-manifolds, see Theorem 2.11 of [7]). Let $\mathcal{E}$ be an eventual identity on $(M_1 \times M_2, \circ, e_1 + e_2)$ and define $\mathcal{E}_k := e_k \circ \mathcal{E}$ for $k \in \{1,2\}$. From (12) $\mathcal{E}_k$ is tangent to M_k at any point of $M_1 \times M_2$. Moreover, $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$, since $e = e_1 + e_2$. We will show that $\mathcal{E}_1$ is a vector field on M_1 (a similar argument shows that $\mathcal{E}_2$ is a vector field on M_2). For this, let Z be a vector field on M_2. Note that

$$L_{\mathcal{E}_1}(\circ)(Z, e_2) = \mathcal{E} \circ L_{e_1}(\circ)(Z, e_2) + e_1 \circ L_{\mathcal{E}}(\circ)(Z, e_2) = 0 \tag{13}$$

because $\mathcal{E}_1 = e_1 \circ \mathcal{E}$, $L_{e_1}(\circ) = 0$ (easy check) and

$$e_1 \circ L_{\mathcal{E}}(\circ)(Z, e_2) = e_1 \circ [e, \mathcal{E}] \circ Z \circ e_2 = 0$$

where we used condition (3) on $\mathcal{E}$ and $e_1 \circ e_2 = 0$. From (13) and $Z = Z \circ e_2$ we get

$$[\mathcal{E}_1, Z] = L_{\mathcal{E}_1}(Z \circ e_2) = [\mathcal{E}_1, Z] \circ e_2 + Z \circ [\mathcal{E}_1, e_2].$$

It follows that $[\mathcal{E}_1, Z]$ is tangent to M_2 at any point of $M_1 \times M_2$. This holds for any vector field Z on M_2 and hence $\mathcal{E}_1$ is a vector field on M_1. Similarly, $\mathcal{E}_2$ is a vector field on M_2. Since $\mathcal{E}$ is invertible on $(M, \circ, e_1 + e_2)$, $\mathcal{E}_1$ is invertible on $(M, \circ_1, e_1)$ and $\mathcal{E}_2$ is invertible on $(M, \circ_2, e_2)$. From

$$[e, \mathcal{E}] = [e_1, \mathcal{E}_1] + [e_2, \mathcal{E}_2]$$

and

$$L_{\mathcal{E}}(\circ)(X, Y) = [e, \mathcal{E}] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M)$$

we get

$$L_{\mathcal{E}_k}(\circ_k)(X, Y) = [e_k, \mathcal{E}_k] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M_k),\ k \in \{1,2\},$$

i.e. $\mathcal{E}_k$ is an eventual identity on the F-manifold $(M_k, \circ_k, e_k)$. Our claim follows. □

By a result of Hertling [7], any F-manifold locally decomposes into a product of irreducible F-manifolds. The decomposition of eventual identities on product F-manifolds into sums of eventual identities on the factors gives a compatibility between our duality for F-manifolds with eventual identities and Hertling's decomposition of F-manifolds, as follows.

Theorem 6. *Let $(M, \circ, e)$ be an F-manifold with irreducible decomposition*

$$(M, \circ, e) \cong (M_1, \circ_1, e_1) \times \cdots \times (M_l, \circ_l, e_l) \tag{14}$$

near a point $p \in M$ and let $\mathcal{E}$ be an eventual identity on $(M, \circ, e)$. Consider the decomposition

$$\mathcal{E} = \mathcal{E}_1 + \cdots + \mathcal{E}_l \tag{15}$$

of $\mathcal{E}$ into a sum of eventual identities $\mathcal{E}_k$ on the factors. Let $(M, *, \mathcal{E}, e)$ be the dual of $(M, \circ, e, \mathcal{E})$ and $(M_k, *_k, \mathcal{E}_k, e_k)$ the dual of $(M_k, \circ_k, e_k, \mathcal{E}_k)$, for any $1 \leqslant k \leqslant l$. Then

$$(M, *, \mathcal{E}) \cong (M_1, *_1, \mathcal{E}_1) \times \cdots \times (M_l, *_l, \mathcal{E}_l) \tag{16}$$

is the irreducible decomposition of the F-manifold $(M, *, \mathcal{E})$ near p.

Proof. The decomposition (15) was proved in Proposition 5 iii). The decomposition (16) follows from (14) and (15). □

We end this section with some more remarks and examples of eventual identities.

Remark 7.

i) Condition (3) which characterizes eventual identities is equivalent to the apparently weaker condition

$$L_{\mathcal{E}}(\circ)(X, Y) = V \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M), \tag{17}$$

for a vector field V. Indeed, if in relation (17) we replace X and Y by e we get $V = L_{\mathcal{E}}(\circ)(e, e)$. On the other hand,

$$L_{\mathcal{E}}(\circ)(e, e) = [\mathcal{E}, e \circ e] - 2[\mathcal{E}, e] \circ e = [e, \mathcal{E}]$$

and hence $V = [e, \mathcal{E}]$, as in (3). In particular, any invertible Euler vector field E of weight d is an eventual identity and $[e, E] = de$.

ii) If $\mathcal{E}$ is an eventual identity on an F-manifold $(M, \circ, e)$, then

$$\left[\mathcal{E}^n, \mathcal{E}^m\right] = (m - n)\mathcal{E}^{m+n-1} \circ [e, \mathcal{E}], \quad \forall m, n \in \mathbb{Z}. \tag{18}$$

The proof is by induction. When $\mathcal{E}$ is Euler and $m, n \geqslant 0$, (18) was proved in [11] (see Theorem 5.6); when $n = -1$ and $m = 0$ (18) was proved in Lemma 4.

iii) Let $(M, \circ, e)$ be a semi-simple F-manifold with canonical coordinates $(u^1, \ldots, u^n)$, i.e.

$$\frac{\partial}{\partial u^i} \circ \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^j}, \quad \forall i, j$$

and

$$e = \frac{\partial}{\partial u^1} + \cdots + \frac{\partial}{\partial u^n}.$$

Any eventual identity is of the form

$$\mathcal{E} = f_1\big(u^1\big)\frac{\partial}{\partial u^1} + \cdots + f_n\big(u^n\big)\frac{\partial}{\partial u^n},$$

where f_i are arbitrary smooth non-vanishing functions depending only on u^i.

iv) Here is an example considered in [7], when the multiplication is not semi-simple. Let $M := \mathbb{R}^2$ with multiplication defined by

$$\frac{\partial}{\partial x^1} \circ \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}, \qquad \frac{\partial}{\partial x^2} \circ \frac{\partial}{\partial x^2} = 0, \quad i \in \{1, 2\}.$$

It can be checked that $\circ$ defines an F-manifold structure and any eventual identity is of the form

$$\mathcal{E} = f_1\big(x^1\big)\frac{\partial}{\partial x^1} + f_2\big(x^1, x^2\big)\frac{\partial}{\partial x^2},$$

where $f_1 = f_1(x^1)$ depends only on x^1 and is non-vanishing, and f_2 is any smooth function.

3. Eventual identities and compatible metrics

The two metrics g and $\tilde{g}$ on a Frobenius manifold have the important property that they form a flat pencil, that is, the metric $g_\lambda^* := g^* + \lambda\tilde{g}^*$ is flat, for all values of λ. This condition results, via the Dubrovin–Novikov theorem, to a bi-Hamiltonian structure. What is important in this construction is not the flatness of the metrics but their compatibility. Curved metrics can, via Ferapontov's extension of the Dubrovin–Novikov theorem [5], define (non-local) Hamiltonian structures but it is the compatibility of two such metrics that will ensure a (non-local) bi-Hamiltonian structure. In this section we construct compatible pairs of metrics on F-manifolds with eventual identities.

We begin by recalling basic definitions and results on compatible pairs of metrics [14]. First we fix the conventions we will use in this and the following sections.

Conventions 8. Let g and $\tilde{g}$ be two metrics on a manifold M, with associated pencil of inverse metrics $g_\lambda^* := g^* + \lambda\tilde{g}^*$ (assumed to be non-degenerate for any λ). We denote by $g : TM \to T^*M$, $X \to g(X)$ and $g^* : T^*M \to TM$, $\alpha \to g^*(\alpha)$ the isomorphisms defined by raising and lowering indices using g and similar notations will be used for the isomorphisms between TM and T^*M defined by $\tilde{g}$ and g_λ. To simplify notations we shall often denote by $X^\flat = \tilde{g}(X)$ the dual 1-form of a vector field X with respect to $\tilde{g}$ (it is important to note that $X^\flat$ is the dual 1-form using $\tilde{g}$ and not g, since the metrics g and $\tilde{g}$ will not play symmetric roles). The Levi-Civita connections of g, g_λ and $\tilde{g}$ will be denoted by ∇, ∇^λ and $\tilde{\nabla}$ respectively; R^g, R^λ and $R^{\tilde{g}}$ and will denote the curvatures of g, g_λ and $\tilde{g}$.

Definition 9. (See [14].)

i) A pair $(g, \tilde{g})$ is called almost compatible if

$$g_\lambda^*\big(\nabla_X^\lambda \alpha\big) = g^*(\nabla_X \alpha) + \lambda \tilde{g}^*(\tilde{\nabla}_X \alpha)$$

for any $X \in \mathcal{X}(M)$, $\alpha \in \Omega^1(M)$ and λ constant.

ii) A pair $(g, \tilde{g})$ is called compatible if $(g, \tilde{g})$ are almost compatible and

$$g_\lambda^*\big(R_{X,Y}^\lambda \alpha\big) = g^*\big(R_{X,Y}^g \alpha\big) + \lambda \tilde{g}^*\big(R_{X,Y}^{\tilde{g}} \alpha\big) \tag{19}$$

for any $X, Y \in \mathcal{X}(M)$, $\alpha \in \Omega^1(M)$ and λ constant.

According to [14] (see also [2] for a shorter proof) the metrics $(g, \tilde{g})$ are almost compatible if and only if the Nijenhuis tensor of $A := g^*\tilde{g} \in \mathrm{End}(TM)$ (or of its inverse $\tilde{g}^* g$) defined by

$$N_A(X, Y) = -[AX, AY] + A\big([AX, Y] + [X, AY]\big) - A^2[X, Y], \quad X, Y \in \mathcal{X}(M)$$

is identically zero. Moreover, according to Theorem 3.1 of [2], if $(g, \tilde{g})$ are almost compatible then $(g, \tilde{g})$ are compatible if and only if one of the following equivalent conditions holds:

$$g^*(\tilde{\nabla}_Y \alpha - \nabla_Y \alpha, \tilde{\nabla}_X \beta - \nabla_X \beta) = g^*(\tilde{\nabla}_X \alpha - \nabla_X \alpha, \tilde{\nabla}_Y \beta - \nabla_Y \beta) \tag{20}$$

or

$$\tilde{g}^*(\tilde{\nabla}_Y \alpha - \nabla_Y \alpha, \tilde{\nabla}_X \beta - \nabla_X \beta) = \tilde{g}^*(\tilde{\nabla}_X \alpha - \nabla_X \alpha, \tilde{\nabla}_Y \beta - \nabla_Y \beta), \tag{21}$$

for any vector fields $X, Y \in \mathcal{X}(M)$ and 1-forms $\alpha, \beta \in \Omega^1(M)$.

We now turn to F-manifolds and we show in Proposition 10 below that an eventual identity on an F-manifold together with a (multiplication) invariant metric $\tilde{g}$ determines a new metric g which is almost compatible with $\tilde{g}$. A metric $\tilde{g}$ on an F-manifold $(M, \circ, e)$ is called invariant if

$$\tilde{g}(X \circ Y, Z) = \tilde{g}(X, Y \circ Z), \quad \forall X, Y, Z \in \mathcal{X}(M)$$

or

$$\tilde{g}(X, Y) = \epsilon(X \circ Y),$$

where $\epsilon = \tilde{g}(e)$ is the coidentity. Thus $\tilde{g}$ is uniquely determined by the coidentity $\epsilon \in \Omega^1(M)$ and invariant metrics on $(M, \circ, e)$ are in bijective correspondence with 1-forms on M.

Proposition 10. *Let $(M, \circ, e, \tilde{g}, \mathcal{E})$ be an F-manifold together with an invariant metric $\tilde{g}$ and eventual identity $\mathcal{E}$. Define a new metric g by*

$$g(X, Y) = \tilde{g}\big(\mathcal{E}^{-1} \circ X, Y\big), \quad \forall X, Y \in \mathcal{X}(M). \tag{22}$$

Then $(g, \tilde{g})$ are almost compatible.

Proof. From (22),

$$g^*\tilde{g}(X) = \mathcal{E} \circ X, \quad \forall X \in TM.$$

Using the F-manifold condition (1) together with the characterization (3) of eventual identities, we get:

$$\begin{aligned}
N_{\mathcal{E}\circ}(X, Y) &= -L_{\mathcal{E}\circ X}(\mathcal{E} \circ Y) + \mathcal{E} \circ \big(L_X(\mathcal{E} \circ Y) - L_Y(\mathcal{E} \circ X)\big) - \mathcal{E}^2 \circ [X, Y] \\
&= -[\mathcal{E} \circ X, \mathcal{E}] \circ Y - [\mathcal{E} \circ X, Y] \circ \mathcal{E} - L_{\mathcal{E}\circ X}(\circ)(\mathcal{E}, Y) \\
&\quad + \mathcal{E} \circ \big([X, \mathcal{E}] \circ Y + \mathcal{E} \circ [X, Y] + L_X(\circ)(\mathcal{E}, Y) - [Y, \mathcal{E}] \circ X\big) \\
&\quad - \mathcal{E}^2 \circ [Y, X] - \mathcal{E} \circ L_Y(\circ)(\mathcal{E}, X) - \mathcal{E}^2 \circ [X, Y] \\
&= L_{\mathcal{E}}(\mathcal{E} \circ X) \circ Y + L_Y(\mathcal{E} \circ X) \circ \mathcal{E} - \mathcal{E} \circ L_X(\circ)(\mathcal{E}, Y) \\
&\quad - X \circ L_{\mathcal{E}}(\circ)(\mathcal{E}, Y) + \mathcal{E} \circ Y \circ [X, \mathcal{E}] + \mathcal{E}^2 \circ [X, Y] \\
&\quad + \mathcal{E} \circ L_X(\circ)(\mathcal{E}, Y) - \mathcal{E} \circ X \circ [Y, \mathcal{E}] - \mathcal{E}^2 \circ [Y, X] \\
&\quad - \mathcal{E} \circ L_Y(\circ)(\mathcal{E}, X) - \mathcal{E}^2 \circ [X, Y] \\
&= L_{\mathcal{E}}(\circ)(\mathcal{E}, X) \circ Y - L_{\mathcal{E}}(\circ)(\mathcal{E}, Y) \circ X \\
&= [e, \mathcal{E}] \circ \mathcal{E} \circ (X \circ Y - Y \circ X) = 0,
\end{aligned}$$

for any vector fields $X, Y \in \mathcal{X}(M)$. Our claim follows. □

When the F-manifold $(M, \circ, e)$ is semi-simple, the pair $(g, \tilde{g})$ of Proposition 10 is semi-simple as well and, being almost compatible, $(g, \tilde{g})$ is automatically compatible [14,2]. Without the semi-simplicity assumption, the pair $(g, \tilde{g})$ is not always compatible. We are going to show that $(g, \tilde{g})$ is compatible (without the semi-simplicity assumption), provided that the coidentity associated to $\tilde{g}$ is closed. To simplify terminology we introduce the following definition.

Definition 11. An almost Riemannian F-manifold is an F-manifold $(M, \circ, e, \tilde{g})$ together with an invariant metric $\tilde{g}$ such that the coidentity $\epsilon \in \Omega^1(M)$ defined by

$$\epsilon(X) := \tilde{g}(e, X), \quad \forall X \in TM$$

is closed.

There is a result of Hertling [7], which states that the closedness of the coidentity ϵ on an F-manifold $(M, \circ, e, \tilde{g})$ with invariant metric is equivalent with the total symmetry of the $(4, 0)$-tensor field

$$(\tilde{\nabla}\circ)(X, Z, Y, V) := \tilde{g}\big(\tilde{\nabla}_X(\circ)(Z, Y), V\big), \tag{23}$$

or to the symmetry in the first two arguments (the symmetry in the last three arguments being a consequence of the invariance of $\tilde{g}$).

Theorem 12. *Let $(M, \circ, e, \tilde{g}, \mathcal{E})$ be an almost Riemannian F-manifold with eventual identity $\mathcal{E}$. Define a new metric g by*

$$g(X, Y) = \tilde{g}\big(\mathcal{E}^{-1} \circ X, Y\big), \quad \forall X, Y \in \mathcal{X}(M). \tag{24}$$

Then $(g, \tilde{g})$ are compatible.

Proof. From Proposition 10, the metrics $(g, \tilde{g})$ are almost compatible. To prove that $(g, \tilde{g})$ are compatible, it is enough to show that (21) is satisfied (see our comments above). The Koszul formula for the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ translated to T^*M gives

$$2\tilde{g}^*(\tilde{\nabla}_Y \alpha, \beta) = -\tilde{g}^*(i_Y d\beta, \alpha) + \tilde{g}^*(i_Y d\alpha, \beta) + Y\tilde{g}^*(\alpha, \beta) - \tilde{g}\big(\big[\tilde{g}^*\alpha, \tilde{g}^*\beta\big], Y\big), \tag{25}$$

for any 1-forms α, β and vector field Y. Using the duality $TM \ni X \to X^\flat \in T^*M$ defined by $\tilde{g}$, we get an induced multiplication on T^*M, also denoted by $\circ$. Replace now in the above relation β by $\tilde{g}g^*(\beta) = \mathcal{E}^\flat \circ \beta$. We get

$$\begin{aligned} 2g^*(\tilde{\nabla}_Y \alpha, \beta) = {} & -d\big(\mathcal{E}^\flat \circ \beta\big)\big(Y, \tilde{g}^*\alpha\big) + g^*(i_Y d\alpha, \beta) + Yg^*(\alpha, \beta) \\ & - \tilde{g}\big(\big[\tilde{g}^*\alpha, \mathcal{E} \circ \tilde{g}^*\beta\big], Y\big). \end{aligned}$$

Combining this relation with the Koszul formula for the Levi-Civita connection ∇ of g on T^*M

$$2g^*(\nabla_Y \alpha, \beta) = -g^*(i_Y d\beta, \alpha) + g^*(i_Y d\alpha, \beta) + Yg^*(\alpha, \beta) - g\big(\big[g^*\alpha, g^*\beta\big], Y\big), \tag{26}$$

we get

$$\begin{aligned} 2g^*(\nabla_Y \alpha - \tilde{\nabla}_Y \alpha, \beta) &= d\big(\mathcal{E}^\flat \circ \beta\big)\big(Y, \tilde{g}^*\alpha\big) - (d\beta)\big(Y, \mathcal{E} \circ \tilde{g}^*\alpha\big) \\ &\quad - g\big(\big[g^*\alpha, g^*\beta\big], Y\big) + \tilde{g}\big(\big[\tilde{g}^*\alpha, \mathcal{E} \circ \tilde{g}^*\beta\big], Y\big) \\ &= d\big(\mathcal{E}^\flat \circ \beta\big)\big(Y, \tilde{g}^*\alpha\big) - (d\beta)\big(Y, \mathcal{E} \circ \tilde{g}^*\alpha\big) \\ &\quad - \tilde{g}\big(\mathcal{E}^{-1} \circ \big[\mathcal{E} \circ \tilde{g}^*\alpha, \mathcal{E} \circ \tilde{g}^*\beta\big] - \big[\tilde{g}^*\alpha, \mathcal{E} \circ \tilde{g}^*\beta\big], Y\big) \end{aligned}$$

where we used (24), $g^*(\alpha) = \mathcal{E} \circ \tilde{g}^*(\alpha)$ and $g^*(\beta) = \mathcal{E} \circ \tilde{g}^*(\beta)$. On the other hand, since the metrics $(g, \tilde{g})$ are almost compatible, the Nijenhuis tensor of $g^*\tilde{g} = \mathcal{E}\circ$ is zero and we obtain

$$\begin{aligned} 2g^*(\nabla_Y \alpha - \tilde{\nabla}_Y \alpha, \beta) &= d\big(\mathcal{E}^\flat \circ \beta\big)\big(Y, \tilde{g}^*\alpha\big) - (d\beta)\big(Y, \mathcal{E} \circ \tilde{g}^*\alpha\big) \\ &\quad + \tilde{g}\big(\mathcal{E} \circ \big[\tilde{g}^*\alpha, \tilde{g}^*\beta\big] - \big[\mathcal{E} \circ \tilde{g}^*\alpha, \tilde{g}^*\beta\big], Y\big). \end{aligned}$$

A similar relation was considered in [2] (see (5.12) in the proof of Proposition 5.10) with the eventual identity $\mathcal{E}$ replaced by a conformal-Killing Euler vector field E. Letting $\alpha := X^\flat$ and $\beta := Z^\flat$ and using a completely similar argument as in [2], one can show that the above relation is equivalent to

$$2g^*\big(\nabla_Y X^\flat - \tilde{\nabla}_Y X^\flat, Z^\flat\big) = (L_{\mathcal{E}}\tilde{g})(X \circ Y, Z) + \tilde{g}\big(\big([e, \mathcal{E}] \circ X - 2\tilde{\nabla}_X \mathcal{E}\big) \circ Y, Z\big). \tag{27}$$

Now, for a vector field V, define a 1-form $(L_{\mathcal{E}}\tilde{g})(V)$ by

$$(L_{\mathcal{E}}\tilde{g})(V)(Z) := (L_{\mathcal{E}}\tilde{g})(V, Z), \quad \forall Z \in \mathcal{X}(M).$$

With this notation,

$$(L_{\mathcal{E}}\tilde{g})(X \circ Y, Z) = (L_{\mathcal{E}}\tilde{g})(X \circ Y)(Z).$$

Since $L_{\mathcal{E}}\tilde{g}$ is multiplication invariant (this follows by taking the Lie derivative with respect to $\mathcal{E}$ of $\tilde{g}(X \circ Y, Z) = \tilde{g}(X, Y \circ Z)$ and using condition (3) on $\mathcal{E}$), we obtain

$$(L_{\mathcal{E}}\tilde{g})(X \circ Y) = X^{\flat} \circ Y^{\flat} \circ (L_{\mathcal{E}}\tilde{g})(e). \tag{28}$$

Denoting $\alpha := X^{\flat}$, from (27) and (28) we get

$$\nabla_Y \alpha - \tilde{\nabla}_Y \alpha = \frac{1}{2} Y^{\flat} \circ \mathcal{E}^{-1,\flat} \circ \left(\left((L_{\mathcal{E}}\tilde{g})(e) + [e, \mathcal{E}]^{\flat}\right) \circ \alpha - 2\tilde{\nabla}_{\tilde{g}^*\alpha}\mathcal{E}^{\flat}\right). \tag{29}$$

Since $\tilde{g}$ is invariant, $\tilde{g}^*$ is also invariant (with respect to the multiplication $\circ$ on T^*M) and relation (29) implies that (21) is satisfied. Being almost compatible, the metrics $(g, \tilde{g})$ are compatible. □

We end this section by making some comments on Theorem 12. Similar results where proved in [2], with the almost Riemannnian F-manifold replaced by a weak $\mathcal{F}$-manifold $(M, \circ, e, \tilde{g}, E)$, i.e. the $C^{\infty}(M)$ bilinear multiplication $\circ$ on TM is commutative, associative, with unit field e, $\tilde{g}$ is an invariant metric, E is an invertible conformal-Killing Euler vector field and the weak symmetry condition

$$(\tilde{\nabla}\circ)(E, Z, Y, V) = (\tilde{\nabla}\circ)(Z, E, Y, V), \quad \forall Y, Z, V \in \mathcal{X}(M) \tag{30}$$

holds; in general, $\circ$ does not satisfy the integrability condition (1), so a weak $\mathcal{F}$-manifold is not always an F-manifold. We are going to show that a weak $\mathcal{F}$-manifold which is also an F-manifold is an almost Riemannian F-manifold. Thus, in the setting of F-manifolds, Theorem 12 extends the statement about the compatibility of metrics in Theorem 5.8 of [2], by replacing the Euler vector field with an eventual identity.

Lemma 13. *Let $(M, \circ, e, \mathcal{E}, \tilde{g})$ be an F-manifold together with an invertible vector field $\mathcal{E}$ and invariant metric $\tilde{g}$. Assume the weak symmetry condition*

$$(\tilde{\nabla}\circ)(\mathcal{E}, Z, Y, V) = (\tilde{\nabla}\circ)(Z, \mathcal{E}, Y, V), \quad \forall Y, Z, V \in \mathcal{X}(M) \tag{31}$$

holds. Then $(M, \circ, e, \tilde{g})$ is an almost Riemannian F-manifold.

Proof. We need to show that the coidentity $\epsilon = \tilde{g}(e)$ is closed. It is known that on any F-manifold $(M, \circ, e, \tilde{g})$ with multiplication $\circ$, unit field e, invariant metric $\tilde{g}$ and coidentity ϵ, the tensor fields $\tilde{\nabla}\circ$ and $d\epsilon$ are related by the following identity (see the proof of Theorem 2.15 of [7]):

$$2(\tilde{\nabla}\circ)(X, Z, Y, V) - 2(\tilde{\nabla}\circ)(Z, X, Y, V) = d\epsilon(Y \circ Z, X \circ V) - d\epsilon(X \circ Y, Z \circ V). \tag{32}$$

Taking $X := \mathcal{E}$ in (32) and using our hypothesis we get

$$d\epsilon(\mathcal{E} \circ Y, Z \circ V) = d\epsilon(Y \circ Z, \mathcal{E} \circ V). \tag{33}$$

With $Z := e$, (33) becomes

$$d\epsilon(\mathcal{E} \circ Y, V) = d\epsilon(Y, \mathcal{E} \circ V). \tag{34}$$

Replacing in (34) V by $V \circ Z$ and using again (33) we get

$$d\epsilon(Y, \mathcal{E} \circ V \circ Z) = d\epsilon(\mathcal{E} \circ Y, V \circ Z) = d\epsilon(Y \circ Z, \mathcal{E} \circ V). \tag{35}$$

Since $\mathcal{E}$ is invertible, relation (35) is equivalent to

$$d\epsilon(Y, Z \circ V) = d\epsilon(Y \circ Z, V), \quad \forall Y, Z, V \in \mathcal{X}(M), \tag{36}$$

i.e. $d\epsilon$ is multiplication invariant. Being skew-symmetric, $d\epsilon = 0$. Our claim follows. $\square$

Finally we show how eventual identities may be used to provide a simple proof of the flatness of (part of) the second structural connection of a Frobenius manifold.

Example 14. Given a Frobenius manifold $(M, \circ, e, E, \tilde{g})$ it is easy to check that the vector field

$$\mathcal{E}_\lambda = E - \lambda e$$

is an eventual identity for all values of the constant λ (we assume that all $\mathcal{E}_\lambda$ are invertible). With this one may define a new metric and multiplication

$$g_\lambda(X, Y) = \tilde{g}\big(\mathcal{E}_\lambda^{-1} \circ X, Y\big),$$
$$X *_\lambda Y = \mathcal{E}_\lambda^{-1} \circ X \circ Y,$$

and, as remarked at the beginning of this section, g_λ is flat for all values of λ. These new structures interpolate between the intersection form and dual multiplication (when $\lambda = 0$) and the Saito metric and original multiplication (as $\lambda \to \infty$). As proved in [7] (Section 9.2), the connection

$$\nabla_X^{\lambda,s} Y = \nabla_X^\lambda Y - sX *_\lambda Y$$

is flat for all λ and s. The connection $\nabla^{\lambda,s}$ is (part of) the second structural connection first introduced, for semi-simple Frobenius manifolds, by Manin and Merkulov [13] and studied further by Hertling [7]. The observation that $\mathcal{E}_\lambda$ is an eventual identity and hence $(M, *_\lambda, \mathcal{E}_\lambda)$ is an F-manifold gives, on using results in [7] (Section 2.5), a simple proof of this result.

4. Duality and Riemannian F-manifolds

Riemannian F-manifolds were first introduced in the literature in [10]. In this section we prove that the class of Riemannian F-manifolds is preserved by the duality between F-manifolds with eventual identities. In the next section we apply this result to the theory of integrable systems.

Definition 15. A Riemannian F-manifold is an F-manifold $(M, \circ, e, \tilde{g})$ together with an invariant metric $\tilde{g}$ such that:

i) the coidentity $\epsilon = \tilde{g}(e) \in \Omega^1(M)$ is closed, i.e. $(M, \circ, e, \tilde{g})$ is an almost Riemannian F-manifold;
ii) the curvature condition

$$Z \circ R^{\tilde{g}}_{V,Y} X + Y \circ R^{\tilde{g}}_{Z,V} X + V \circ R^{\tilde{g}}_{Y,Z} X = 0 \tag{37}$$

is satisfied, for any $X, Y, Z, V \in \mathcal{X}(M)$.

Our main result in this section is the following theorem.

Theorem 16. *Let $(M, \circ, e, \tilde{g}, \mathcal{E})$ be an F-manifold with invariant metric $\tilde{g}$ and eventual identity $\mathcal{E}$. Define a second metric g by*

$$g(X, Y) = \tilde{g}\big(\mathcal{E}^{-1} \circ X, Y\big), \quad \forall X, Y \in \mathcal{X}(M) \tag{38}$$

and let $(M, *, \mathcal{E}, e)$ be the dual of $(M, \circ, e, \mathcal{E})$. Then $(M, \circ, e, \tilde{g})$ is a Riemannian F-manifold if and only if $(M, *, \mathcal{E}, g)$ is a Riemannian F-manifold.

Proof. From (38), the coidentities of $(M, \circ, e, \tilde{g})$ and $(M, *, \mathcal{E}, g)$ coincide. Thus $(M, \circ, e, \tilde{g})$ is an almost Riemannian F-manifold if and only if $(M, *, \mathcal{E}, g)$ is an almost Riemannian F-manifold.

Assume now that $(M, \circ, e, \tilde{g})$ is a Riemannian F-manifold. By our comments from the previous section, the tensor field $\tilde{\nabla}\circ$ is totally symmetric. With the conventions from the proof of Theorem 12, the total symmetry of $\tilde{\nabla}\circ$ and relation (29), together with an easy curvature computation show that the curvatures of g and $\tilde{g}$ on T^*M are related by

$$R^{g}_{X,Y}\alpha = R^{\tilde{g}}_{X,Y}\alpha + Q(\alpha, Y) \circ X^{\flat} - Q(\alpha, X) \circ Y^{\flat}, \tag{39}$$

where

$$Q(\alpha, X) := \mathcal{S}\big(\mathcal{S}(\alpha) \circ X^{\flat}\big) - \tilde{\nabla}_X(\mathcal{S})(\alpha), \quad \forall \alpha \in T^*M,\ \forall X \in TM$$

and $\mathcal{S} \in \mathrm{End}(T^*M)$ is defined by

$$\mathcal{S}(\alpha) := \frac{1}{2}\mathcal{E}^{-1,\flat} \circ \Big(\big((L_{\mathcal{E}}\tilde{g})(e) + [e, \mathcal{E}]^{\flat}\big) \circ \alpha - 2\tilde{\nabla}_{\tilde{g}^*\alpha}\mathcal{E}^{\flat}\Big). \tag{40}$$

(Recall that TM and T^*M are identified using $\tilde{g}$ and $\circ$ above denotes the induced multiplication on T^*M.) Since $(M, \circ, e, \tilde{g})$ is a Riemannian F-manifold, relation (37) holds. Translated to T^*M, it gives

$$Z^\flat \circ R^{\tilde{g}}_{V,Y}\alpha + Y^\flat \circ R^{\tilde{g}}_{Z,V}\alpha + V^\flat \circ R^{\tilde{g}}_{Y,Z}\alpha = 0, \tag{41}$$

for any vector fields Y, Z and V and covector α. Using (39), relation (41) becomes

$$Z^\flat \circ R^{g}_{V,Y}\alpha + Y^\flat \circ R^{g}_{Z,V}\alpha + V^\flat \circ R^{g}_{Y,Z}\alpha = 0. \tag{42}$$

Take in (42) $\alpha := g(X)$. Note that

$$Z^\flat \circ R^{g}_{V,Y}\alpha = Z^\flat \circ g\big(R^{g}_{V,Y}X\big) = Z^\flat \circ \mathcal{E}^{-1,\flat} \circ \big(R^{g}_{V,Y}X\big)^\flat$$

and similarly for $Y^\flat \circ R^{g}_{Z,V}\alpha$ and $V^\flat \circ R^{g}_{Y,Z}\alpha$. On TM relation (42) becomes

$$\mathcal{E}^{-1} \circ \big(Z \circ R^{g}_{V,Y}X + Y \circ R^{g}_{Z,V}X + V \circ R^{g}_{Y,Z}X\big) = 0 \tag{43}$$

for any vector fields X, Y, Z, V, or

$$Z * R^{g}_{V,Y}X + Y * R^{g}_{Z,V}X + V * R^{g}_{Y,Z}X = 0, \tag{44}$$

from the definition of $*$. We proved that $(M, *, \mathcal{E}, g)$ is a Riemannian F-manifold. Our claim follows. $\square$

5. Applications to integrable systems

There is a close relationship between F-manifolds and the theory of integrable systems of hydrodynamic type. In particular we draw together various results of [10] into the following theorem.

Theorem 17. *Consider an almost Riemannian F-manifold $(M, \circ, e, \tilde{g})$. If $\tilde{X}_1$ and $\tilde{X}_2$ are two vector fields which satisfy the condition*

$$(\tilde{\nabla}_Z \tilde{X}_i) \circ V = (\tilde{\nabla}_V \tilde{X}_i) \circ Z, \quad \forall V, Z \in \mathcal{X}(M),\ i \in \{1, 2\} \tag{45}$$

then the associated flows

$$U_t = \tilde{X}_1 \circ U_x,$$
$$U_\tau = \tilde{X}_2 \circ U_x$$

commute. Moreover, for arbitrary vector fields $Y, V, Z \in \mathcal{X}(M)$ the identity

$$Z \circ R^{\tilde{g}}_{V,Y}\tilde{X} + Y \circ R^{\tilde{g}}_{Z,V}\tilde{X} + V \circ R^{\tilde{g}}_{Y,Z}\tilde{X} = 0$$

holds for any solution $\tilde{X}$ of (45).

By twisting solutions $\tilde{X}$ of (45) by an eventual identity one may derive the dual, or twisted, version of the above theorem.

Lemma 18. *Let $(M, \circ, e, \tilde{g})$ be an almost Riemannian F-manifold and $\tilde{X} \in \mathcal{X}(M)$ a vector field such that*

$$(\tilde{\nabla}_Y \tilde{X}) \circ V = (\tilde{\nabla}_V \tilde{X}) \circ Y, \quad \forall Y, V \in \mathcal{X}(M). \tag{46}$$

Let $\mathcal{E}$ be an eventual identity on $(M, \circ, e)$ and $(M, *, \mathcal{E}, g)$ the dual almost Riemannian F-manifold, like in Theorem 16. *Then $X = \tilde{X} \circ \mathcal{E}$ satisfies the dual equation*

$$(\nabla_Y X) * V = (\nabla_V X) * Y, \quad \forall Y, V \in \mathcal{X}(M). \tag{47}$$

Proof. Recall, from relation (29), that

$$\nabla_Y \alpha = \tilde{\nabla}_Y \alpha + Y^\flat \circ \mathcal{S}(\alpha), \quad \forall Y \in TM, \ \forall \alpha \in \Omega^1(M) \tag{48}$$

where $\mathcal{S}(\alpha)$ is given by (40). In (48) let $\alpha := \tilde{X}^\flat = g(\tilde{X} \circ \mathcal{E})$. Relation (48) becomes

$$g\big(\nabla_Y(\tilde{X} \circ \mathcal{E})\big) = \tilde{\nabla}_Y \tilde{X}^\flat + Y^\flat \circ \mathcal{S}\big(\tilde{X}^\flat\big). \tag{49}$$

Applying g^* to (49) and using $g^*\tilde{g}(X) = \mathcal{E} \circ X$ for any X, we get

$$\nabla_Y(\tilde{X} \circ \mathcal{E}) = \mathcal{E} \circ \tilde{\nabla}_Y \tilde{X} + \mathcal{E} \circ Y \circ \tilde{g}^*\big(\mathcal{S}\big(\tilde{X}^\flat\big)\big). \tag{50}$$

From (50) and the definition of $*$ we get

$$\nabla_Y(\tilde{X} \circ \mathcal{E}) * V = (\tilde{\nabla}_Y \tilde{X}) \circ V + Y \circ V \circ \tilde{g}^*\big(\mathcal{S}\big(\tilde{X}^\flat\big)\big),$$

which, from (46), is symmetric in Y and V. Relation (47) is satisfied. $\square$

Thus we obtain dual flow equations

$$U_t = X_1 * U_x,$$
$$U_\tau = X_2 * U_x$$

from vector fields $\tilde{X}_1, \tilde{X}_2 \in \mathcal{X}(M)$ satisfying (45) by twisting by an eventual identity. Moreover, by Theorem 17 and the above lemma, the dual curvature condition also holds.

This duality, or twisting, by an eventual identity gives a geometric form of certain well-known arguments from the theory of integrable systems of hydrodynamic type which originate in the work of Tsarev [19]. Recall that in the semi-simple case the basic equation $U_t = \tilde{X} \circ U_x$ reduces to diagonal form

$$u^i_t = \tilde{X}^i(\mathbf{u}) u^i_x$$

so the components of $\tilde{X}$ become the characteristic velocities of the quasilinear system. Eq. (45) reduces to Tsarev's equation [19]

$$\frac{\partial}{\partial u^i}\log\sqrt{\tilde{g}_{jj}}=\frac{\partial_i\tilde{X}^j}{\tilde{X}^i-\tilde{X}^j},\quad i\neq j. \tag{51}$$

The integrability conditions for this system form the so-called semi-Hamiltonian conditions, which in turn are the coordinate form of (37).

Solutions of (51) possess a functional freedom: if $\tilde{g}_{ii}(\mathbf{u})$ is a solution so is $\tilde{g}_{ii}(\mathbf{u})/f_i(u^i)$. This functional freedom can now be reinterpreted, via Remark 7 iii) on the form of eventual identities in the semi-simple case, as the dual version of the theory. Thus any two solutions of Tsarev's equation are connected by an eventual identity. Also since the f_i are arbitrary, one may replace them by $f_i \to f_i+\lambda$ for any constant λ. Thus one recovers the pencil property $g^*_\lambda = g^*+\lambda\tilde{g}^*$ and hence, by Theorem 12, a compatible pair of metrics and (non-local) bi-Hamiltonian structures (this last stage, from almost compatible to compatible being automatic in the semi-simple case).

In applications, where one is interested in finding bi-Hamiltonian structures for a specific system of equations, one tries to find a suitable eventual identity so that the metric g has simple curvature properties, such as flatness or constant curvature. If flat, one arrives, via the original Dubrovin–Novikov theorem, at a local Hamiltonian structure. The simplest case is where both metrics are flat, and hence form a flat pencil and a local bi-Hamiltonian structure. With extra conditions one can arrive at a Frobenius manifold [4].

6. Duality and tt^*-geometry

An holomorphic F-manifold is a complex manifold M together with an $\mathcal{O}_M$-bilinear, associative, commutative multiplication $\circ$ with unit field on the sheaf of holomorphic vector fields, satisfying the F-manifold condition (1). An holomorphic vector field $\mathcal{E}$ on M is called invertible if there is a vector field $\mathcal{E}^{-1}$, also holomorphic, such that $\mathcal{E}\circ\mathcal{E}^{-1}=e$ where e is the unit field. An eventual identity is an holomorphic invertible vector field $\mathcal{E}$ such that the multiplication (2) defines a new (holomorphic) F-manifold structure on M. Theorem 3 holds also in the holomorphic setting.

In Sections 6.1 and 6.2 we consider the interactions between the duality for F-manifolds with eventual identities and harmonic Higgs bundles, $DChk$-structures and weak CV-structures. In Section 6.3 we discuss the simplest class of examples – when the underlying F-manifold is semi-simple and both metric and real structure are diagonal. First we fix our conventions in the holomorphic setting.

Conventions 19. In this section M will denote a complex manifold, considered as a smooth manifold together with an integrable complex structure J. Its real tangent bundle will be denoted TM. The sheaf of smooth real vector fields on (M,J) will be denoted as always by $\mathcal{X}(M)$, the sheaf of vector fields of type $(1,0)$ by $\mathcal{T}^{1,0}_M$, the sheaf of holomorphic vector fields by $\mathcal{T}_M$ and the sheaf of $(1,0)$-forms with values in a complex vector bundle $E\to M$ by $\Omega^{1,0}(M,E)$. The multiplication of an holomorphic F-manifold structure on M will be extended by $C^\infty(M)$-bilinearity to the holomorphic tangent bundle $T^{1,0}M$, or to the complexified tangent bundle $T_{\mathbb{C}}M=TM\otimes\mathbb{C}$ (such that if $X\in T^{0,1}M$ then $X\circ Y=Y\circ X=0$ for any $Y\in T_{\mathbb{C}}M$). The Hermitian metrics we will consider are non-degenerate, but not necessarily positive definite.

6.1. Duality, harmonic Higgs bundles and DChk-structures

Following [1,6,15] we recall basic notions from the theory of tt^*-geometry.

Definition 20.

i) A pair $(\tilde{g}, \tilde{h})$ formed by a complex bilinear, non-degenerate symmetric form $\tilde{g}$ and a Hermitian metric $\tilde{h}$ on $T^{1,0}M$ is called compatible if the Chern connection $\tilde{D}$ of the holomorphic Hermitian vector bundle $(T^{1,0}M, \tilde{h})$ preserves $\tilde{g}$, i.e. $\tilde{D}_X\tilde{g} = 0$, for any $X \in TM$.

ii) Let $\tilde{h}$ be a Hermitian metric and $\circ$ a $C^\infty(M)$ bilinear, commutative, associative, multiplication on $T^{1,0}M$, with unit field. Define a Higgs field $\tilde{C} \in \Omega^{1,0}(M, \mathrm{End}(T^{1,0}M))$ by

$$\tilde{C}_X Y := X \circ Y, \quad \forall X, Y \in T^{1,0}M.$$

The Hermitian metric $\tilde{h}$ on the Higgs bundle $(T^{1,0}M, \tilde{C})$ is called harmonic (and $(T^{1,0}M, \tilde{C}, \tilde{h})$ is a harmonic Higgs bundle) if $\tilde{C}_X Y \in \mathcal{T}_M$, for any $X, Y \in \mathcal{T}_M$ and the tt^*-equations

$$\left(\partial^{\tilde{D}}\tilde{C}\right)_{X,Y} := \tilde{D}_X(\tilde{C}_Y) - \tilde{D}_Y(\tilde{C}_X) - \tilde{C}_{[X,Y]} = 0 \tag{52}$$

and

$$R^{\tilde{D}}_{X,\bar{Y}} + \left[\tilde{C}_X, \tilde{C}^\flat_{\bar{Y}}\right] = 0 \tag{53}$$

are satisfied, for any $X, Y \in \mathcal{T}^{1,0}_M$. Above $R^{\tilde{D}}$ denotes the curvature of the Chern connection $\tilde{D}$ of $(T^{1,0}M, \tilde{h})$ and $\tilde{C}^\flat \in \Omega^{0,1}(M, \mathrm{End}(T^{1,0}M))$ is the adjoint of $\tilde{C}$ with respect to $\tilde{h}$, i.e.

$$\tilde{h}(\tilde{C}_X Y, Z) = \tilde{h}\left(Y, \tilde{C}^\flat_{\bar{X}} Z\right), \quad \forall X, Y, Z \in T^{1,0}M.$$

iii) Let $(T^{1,0}M, \tilde{C}, \tilde{h})$ be a harmonic Higgs bundle and $\tilde{k}$ a real structure on $T^{1,0}M$ (i.e. a complex anti-linear involution of $T^{1,0}M$) such that the complex bilinear form

$$\tilde{g}(X, Y) := \tilde{h}(X, \tilde{k}Y)$$

on $T^{1,0}M$ is symmetric and (multiplication) invariant. The data $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k})$ is called a $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$-structure if the pair $(\tilde{g}, \tilde{h})$ is compatible (or $\tilde{D}_X\tilde{k} = 0$ for any $X \in TM$).

A harmonic Higgs bundle $(T^{1,0}M, \tilde{C}, \tilde{h})$ has an associated pencil of flat connections

$$\tilde{D}^z := \tilde{D} + \frac{1}{z}\tilde{C} + z\tilde{C}^\flat. \tag{54}$$

The flatness property of this pencil encodes the entire geometry of the harmonic Higgs bundle [6].

For the remaining part of this section we fix an holomorphic F-manifold $(M, \circ, e)$ with associated Higgs field

$$\tilde{C}_X Y := X \circ Y,$$

together with an eventual identity $\mathcal{E}$, Hermitian metric $\tilde{h}$ and real structure $\tilde{k}$ on $T^{1,0}M$ such that the complex bilinear form

$$\tilde{g}(X,Y) := \tilde{h}(X, \tilde{k}Y)$$

on $T^{1,0}M$ is symmetric and invariant. Let

$$X * Y := X \circ Y \circ \mathcal{E}^{-1} \tag{55}$$

be the dual multiplication, with associated Higgs field

$$C_X Y := X * Y = \tilde{C}_{\mathcal{E}^{-1}} \tilde{C}_X Y. \tag{56}$$

Assume that the inverse $\mathcal{E}^{-1}$ has a square root $\mathcal{E}^{-1/2}$ and define a new Hermitian metric

$$h(X,Y) := \tilde{h}\big(\mathcal{E}^{-1/2} \circ X, \mathcal{E}^{-1/2} \circ Y\big) \tag{57}$$

and a new real structure

$$k(X) := \mathcal{E}^{1/2} \circ \tilde{k}\big(\mathcal{E}^{-1/2} \circ X\big)$$

on $T^{1,0}M$. It is straightforward to check that

$$g(X,Y) := h(X, kY) = \tilde{g}\big(\mathcal{E}^{-1/2} \circ X, \mathcal{E}^{-1/2} \circ Y\big). \tag{58}$$

In particular, g is symmetric, complex bilinear and invariant.

While in the smooth case it was not immediately clear that compatibility is preserved under twisting with eventual identities, the analogous statement in the holomorphic setting comes for free (and in fact holds under the weaker assumption that $\mathcal{E}$ is holomorphic invertible, not necessarily an eventual identity).

Lemma 21. *If the pair $(\tilde{g}, \tilde{h})$ is compatible, then the pair (g, h) is also compatible.*

Proof. From (57) together with $\mathcal{E}$-holomorphic invertible, the Chern connections D and $\tilde{D}$ of $(T^{1,0}M, h)$ and $(T^{1,0}M, \tilde{h})$ respectively are related by

$$D_X Z := \mathcal{E}^{1/2} \circ \tilde{D}_X\big(\mathcal{E}^{-1/2} \circ Z\big), \quad \forall X \in \mathcal{X}(M),\ Z \in \mathcal{T}^{1,0}_M. \tag{59}$$

From (58) and (59), $\tilde{D}\tilde{g} = 0$ if and only if $Dg = 0$. $\square$

Note that if M is a Frobenius manifold with Euler vector field E then the choice $\mathcal{E} = E$ results in a compatible pair (g, h) with certain special properties. The metric g is the intersection form of the manifold, and hence is flat. Thus there exists a distinguished coordinate system of so-called flat coordinates in which the components of g are constant. The metric h is then a natural Hermitian metric defined on the complement of the classical discriminant Σ of the manifold.

We now state our main result from this section.

Theorem 22.

i) *Assume that* $\partial^{\tilde{D}}\tilde{C}=0$. *Then* $\partial^{D}C=0$ *if and only if for any* $X, Y, Z \in \mathcal{T}_M^{1,0}$,

$$\tilde{D}_X(\mathcal{E}\circ Y\circ Z)-\tilde{D}_Y(\mathcal{E}\circ X\circ Z)=\mathcal{E}\circ\big(\tilde{D}_X(Y\circ Z)-\tilde{D}_Y(X\circ Z)\big). \tag{60}$$

ii) *Assume that for any* $X, Y \in \mathcal{T}_M^{1,0}$,

$$R^{\tilde{D}}_{X,\bar{Y}}+\big[\tilde{C}_X,\tilde{C}^{\flat}_{\bar{Y}}\big]=0. \tag{61}$$

Then the same relation holds with $\tilde{D}$ *replaced by* D, $\tilde{C}$ *replaced by* C *and* $\tilde{C}^{\flat}$ *replaced by the adjoint* $C^{\flat}$ *of* C *with respect to the Hermitian metric* h *if and only if, for any* $X, Y \in T^{1,0}M$,

$$[\tilde{C}_X,\tilde{k}\tilde{C}_Y\tilde{k}]=[\tilde{C}_{\mathcal{E}^{-1}\circ X},\tilde{k}\tilde{C}_{\mathcal{E}^{-1}\circ Y}\tilde{k}]. \tag{62}$$

iii) *If* $(T^{1,0}M,\tilde{C},\tilde{h})$ *is a harmonic Higgs bundle (respectively,* $(T^{1,0}M,\tilde{C},\tilde{h},\tilde{k})$ *is a* $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$*-structure) then* $(T^{1,0}M,C,h)$ *is a harmonic Higgs bundle (respectively,* $(T^{1,0}M,C,h,k)$ *is a* $DChk$*-structure) if and only if both* (60) *and* (62) *are satisfied.*

Proof. From (59), the connections D and $\tilde{D}$ are related on $\mathrm{End}(T^{1,0}M)$ by

$$D_XT=\tilde{D}_XT+\big[\tilde{C}_{\mathcal{E}^{1/2}}\tilde{D}_X(\tilde{C}_{\mathcal{E}^{-1/2}}),T\big], \tag{63}$$

where $X\in TM$ and T is any section of $\mathrm{End}(T^{1,0}M)$. From (56), (63) and the tt^*-equation $\partial^{\tilde{D}}\tilde{C}=0$, we get

$$\begin{aligned}\big(\partial^DC\big)_{X,Y}&=\tilde{C}_{\mathcal{E}^{1/2}}\big(\tilde{D}_X(\tilde{C}_{\mathcal{E}^{-3/2}})\tilde{C}_Y-\tilde{C}_{\mathcal{E}^{-1}}\tilde{C}_Y\tilde{D}_X(\tilde{C}_{\mathcal{E}^{-1/2}})\big)\\&\quad-\tilde{C}_{\mathcal{E}^{1/2}}\big(\tilde{D}_Y(\tilde{C}_{\mathcal{E}^{-3/2}})\tilde{C}_X-\tilde{C}_{\mathcal{E}^{-1}}\tilde{C}_X\tilde{D}_Y(\tilde{C}_{\mathcal{E}^{-1/2}})\big),\end{aligned}$$

for any $X, Y\in\mathcal{T}_M^{1,0}$. On the other hand,

$$\tilde{C}_X\tilde{D}_Y(\tilde{C}_{\mathcal{E}^{-1/2}})-\tilde{C}_Y\tilde{D}_X(\tilde{C}_{\mathcal{E}^{-1/2}})=\tilde{D}_Y(\tilde{C}_{\mathcal{E}^{-1/2}})\tilde{C}_X-\tilde{D}_X(\tilde{C}_{\mathcal{E}^{-1/2}})\tilde{C}_Y \tag{64}$$

which follows by applying $\tilde{D}_X$ to the equality $\tilde{C}_Y\tilde{C}_{\mathcal{E}^{-1/2}}=\tilde{C}_{\mathcal{E}^{-1/2}}\tilde{C}_Y$, skew-symmetrizing in X and Y and using $\partial^{\tilde{D}}\tilde{C}=0$. From (64) and the above expression of ∂^DC we get:

$$\big(\partial^DC\big)_{X,Y}=\tilde{C}_{\mathcal{E}^{1/2}}\big(\tilde{D}_X(\tilde{C}_{\mathcal{E}^{-1}})\tilde{C}_Y-\tilde{D}_Y(\tilde{C}_{\mathcal{E}^{-1}})\tilde{C}_X\big)\tilde{C}_{\mathcal{E}^{-1/2}},$$

which readily implies the first claim (easy check). For the second claim, assume that (61) holds. We need to show that (62) is equivalent to the tt^*-equation

$$R^D_{X,\bar{Y}}+\big[C_X,C^{\flat}_{\bar{Y}}\big]=0,\quad\forall X,Y\in T^{1,0}M. \tag{65}$$

From (59),

$$R^{D}_{X,\bar{Y}} = \tilde{C}_{\mathcal{E}^{1/2}} R^{\tilde{D}}_{X,\bar{Y}} \tilde{C}_{\mathcal{E}^{-1/2}}, \quad \forall X, Y \in T^{1,0}M. \tag{66}$$

Since $\tilde{g}$ is $\circ$-invariant,

$$\tilde{C}^{\flat}_{\bar{Y}} = \tilde{k}\tilde{C}_Y\tilde{k}, \quad \forall Y \in T^{1,0}M \tag{67}$$

and similarly

$$C^{\flat}_{\bar{Y}} = kC_Yk, \quad \forall Y \in T^{1,0}M \tag{68}$$

because g is $*$-invariant. Using the definition of C and k, (68) becomes

$$C^{\flat}_{\bar{Y}} = \tilde{C}_{\mathcal{E}^{1/2}}\tilde{k}\tilde{C}_{Y\circ\mathcal{E}^{-1}}\tilde{k}\tilde{C}_{\mathcal{E}^{-1/2}}, \quad \forall Y \in T^{1,0}M. \tag{69}$$

The equivalence of (62) and (65) is a consequence of (61), (66) and (69). This proves the second claim. The third claim is now trivial, from our considerations above (see also Lemma 21). □

We remark that condition (60) on the eventual identity is invariant under the duality of Theorem 3 ii). The following simple result holds.

Proposition 23. *Let $(M, *, \mathcal{E}, e)$ be the dual of $(M, \circ, e, \mathcal{E})$. If the eventual identity $\mathcal{E}$ of $(M, \circ, e)$ satisfies*

$$\tilde{D}_X(\mathcal{E}\circ Y\circ Z) - \tilde{D}_Y(\mathcal{E}\circ X\circ Z) = \mathcal{E}\circ\big(\tilde{D}_X(Y\circ Z) - \tilde{D}_Y(X\circ Z)\big) \tag{70}$$

then the eventual identity e of $(M, *, \mathcal{E})$ satisfies the dual condition

$$D_X(e * Y * Z) - D_Y(e * X * Z) = e * \big(D_X(Y * Z) - D_Y(X * Z)\big), \tag{71}$$

for any $X, Y, Z \in \mathcal{T}^{1,0}_M$.

Proof. Straightforward computation, which uses (55) and (59). □

6.2. Duality and weak CV-structures

A CV-structure on the holomorphic tangent bundle of a complex manifold M is a $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$-structure together with two endomorphisms $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{Q}}$ of $T^{1,0}M$, satisfying some additional compatibility conditions. In particular, the endomorphism $\tilde{\mathcal{Q}}$ is Hermitian with respect to $\tilde{h}$ and, as it turns out, $\tilde{\mathcal{U}} = \tilde{C}_E$, where E is an Euler vector field of weight one on the F-manifold underlying the $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$-structure.

It is immediately clear that the class of CV-structures is not preserved by the duality for F-manifolds with eventual identities. The reason is that if E is an invertible Euler vector field on an F-manifold $(M, \circ, e)$, then e is not Euler for the dual F-manifold $(M, *, E)$. In Section 6.2.1 we define CV-structures in a weaker sense, with the Euler vector field replaced by an eventual identity. In Section 6.2.2 we prove that the class of weak CV-structures so defined is preserved by our duality for F-manifolds with eventual identities, provided that the eventual identity satisfies conditions (60) and (62) of Theorem 22.

6.2.1. Weak CV-structures

We begin by recalling basic definitions and results about CV-structures on the holomorphic tangent bundle of a complex manifold. Our treatment of CV-structures follows closely [6], where more details and proofs can be found. It is worth pointing out some differences between our conventions and those used in [6]. While we use the generic notation $\tilde{C}$ for a Higgs field and $\tilde{C}^{\flat}$ for its adjoint with respect to a Hermitian metric, the general notation in [6] for a Higgs field is C (which in our conventions is the dual Higgs field) and $\tilde{C}$ denotes its adjoint with respect to a Hermitian metric. Moreover, in our conventions the Higgs field $\tilde{C}$ is related to the associated multiplication $\circ$ on the tangent bundle by $\tilde{C}_X Y = X \circ Y$, while in [6] $C_X Y = -X \circ Y$. Hopefully these differences will not generate confusions.

Definition 24. (See [6].) A CV-structure is a $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$-structure $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k})$ together with two endomorphisms $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{Q}}$ of $T^{1,0}M$ such that the following conditions hold:

i) for any $X \in T^{1,0}M$, $[\tilde{C}_X, \tilde{\mathcal{U}}] = 0$;
ii) $\tilde{D}_{\bar{X}}\tilde{\mathcal{U}} = 0$ for any $X \in T^{1,0}M$, i.e. if $Z \in \mathcal{T}_M$ then also $\tilde{\mathcal{U}}(Z) \in \mathcal{T}_M$;
iii) the $(1,0)$-part of $\tilde{D}\tilde{\mathcal{U}}$ is given by

$$\tilde{D}_X\tilde{\mathcal{U}} + [\tilde{C}_X, \tilde{\mathcal{Q}}] - \tilde{C}_X = 0, \quad \forall X \in T^{1,0}M; \tag{72}$$

iv) $\tilde{\mathcal{Q}}$ is Hermitian with respect to $\tilde{h}$; moreover, $\tilde{\mathcal{Q}} + \tilde{k}\tilde{\mathcal{Q}}\tilde{k} = 0$, or, equivalently, $\tilde{\mathcal{Q}}$ is skew-symmetric with respect to the complex bilinear form $\tilde{g}$, defined as usual by $\tilde{g}(X,Y) = \tilde{h}(X, \tilde{k}Y)$, for any $X, Y \in T^{1,0}M$;
v) the $(1,0)$-part of $\tilde{D}\tilde{\mathcal{Q}}$ is given by

$$\tilde{D}_X\tilde{\mathcal{Q}} - [\tilde{C}_X, \tilde{k}\tilde{\mathcal{U}}\tilde{k}] = 0, \quad \forall X \in T^{1,0}M. \tag{73}$$

Let $\circ$ be the multiplication on $T^{1,0}M$, related to the Higgs field $\tilde{C}$ by $X \circ Y := \tilde{C}_X Y$, for any $X, Y \in \mathcal{T}_M^{1,0}$ and denote by $e \in \mathcal{T}_M$ its unit field. Then $(M, \circ, e)$ is an F-manifold (this is a consequence of the tt^*-equation $\partial^{\tilde{D}}\tilde{C} = 0$, see Lemma 4.3 of [6] for the proof). From i), $\tilde{\mathcal{U}}$ is the multiplication by a vector field $E = \tilde{\mathcal{U}}(e) \in \mathcal{T}^{1,0}M$. Condition ii) together with $e \in \mathcal{T}_M$ imply that E is holomorphic and condition (72) with $\tilde{\mathcal{U}} = \tilde{C}_E$ implies that E is an Euler vector field of weight one for $(M, \circ, e)$ (again, by Lemma 4.3 of [6]). In particular $[e, E] = e$.

We now define the more general notion of weak CV-structure.

Definition 25. A weak CV-structure is a $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$-structure $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k})$ together with two endomorphisms $\tilde{\mathcal{U}} = \tilde{C}_{\mathcal{E}}$ (where $\mathcal{E} \in \mathcal{T}_M$) and $\tilde{\mathcal{Q}}$ of $T^{1,0}M$, satisfying all conditions of Definition 24, except that (72) is replaced by the weaker condition

$$\tilde{D}_X\tilde{\mathcal{U}} + [\tilde{C}_X, \tilde{\mathcal{Q}}] - \tilde{C}_{[e,\mathcal{E}]}\tilde{C}_X = 0, \quad \forall X \in T^{1,0}M. \tag{74}$$

While a CV-structure determines a preferred Euler vector field on the underlying F-manifold, a weak CV-structure determines a holomorphic vector field $\mathcal{E}$ which satisfies the weaker condition (75), see below. In particular, if $\mathcal{E}$ is invertible then it is an eventual identity.

Lemma 26. *Let $(M, \circ, e)$ be an F-manifold and $\tilde{D}$ a connection on $T^{1,0}M$ such that $\partial^{\tilde{D}}\tilde{C} = 0$, where $\tilde{C}_X Y = X \circ Y$ is the Higgs field. Let $\mathcal{E}$ be a vector field of type $(1,0)$ on M.*

i) *Assume that*

$$L_{\mathcal{E}}(\circ)(X, Y) = [e, \mathcal{E}] \circ X \circ Y, \quad \forall X, Y \in \mathcal{T}^{1,0}_M. \tag{75}$$

Then

$$\tilde{D}_X(\tilde{C}_{\mathcal{E}}) + [\tilde{C}_X, \tilde{D}_{\mathcal{E}} - L_{\mathcal{E}}] - \tilde{C}_{[e,\mathcal{E}]}\tilde{C}_X = 0, \quad \forall X \in \mathcal{T}^{1,0}_M. \tag{76}$$

ii) *Conversely, assume that*

$$\tilde{D}_X(\tilde{C}_{\mathcal{E}}) + [\tilde{C}_X, \tilde{\mathcal{Q}}] - \tilde{C}_{[e,\mathcal{E}]}\tilde{C}_X = 0, \quad \forall X \in \mathcal{T}^{1,0}_M, \tag{77}$$

for an endomorphism $\tilde{Q}$ of $T^{1,0}M$. Then $\mathcal{E}$ satisfies (75) *and $\tilde{Q}$ is equal to $\tilde{D}_{\mathcal{E}} - L_{\mathcal{E}}$ up to addition with $\tilde{C}_Z$ for $Z \in \mathcal{T}^{1,0}_M$.*

Proof. Assume that (75) holds. Then, for any $X \in \mathcal{T}^{1,0}_M$,

$$\begin{aligned}\tilde{D}_X(\tilde{C}_{\mathcal{E}}) + [\tilde{C}_X, \tilde{D}_{\mathcal{E}} - L_{\mathcal{E}}] &= \tilde{D}_X(\tilde{C}_{\mathcal{E}}) - \tilde{D}_{\mathcal{E}}(\tilde{C}_X) + [L_{\mathcal{E}}, \tilde{C}_X] \\ &= \tilde{C}_{[X,\mathcal{E}]} + L_{\mathcal{E}}(X\circ) = \tilde{C}_{[e,\mathcal{E}]\circ X},\end{aligned}$$

where in the second equality we used the tt^*-equation $\partial^{\tilde{D}}\tilde{C} = 0$ and in the third equality we used condition (75). The first claim follows. We now prove the second claim. As already mentioned above, if $[e, \mathcal{E}] = e$ then (77) implies that $\mathcal{E}$ is Euler of weight one, by Lemma 4.3 of [6]. Without this additional assumption, the same argument shows that (77) implies (75). Therefore, (76) holds as well and $\tilde{\mathcal{Q}} - \tilde{D}_{\mathcal{E}} + L_{\mathcal{E}}$ commutes with $\tilde{C}_X$ for any $X \in T^{1,0}M$. Thus $\tilde{\mathcal{Q}} - \tilde{D}_{\mathcal{E}} + L_{\mathcal{E}}$ is the multiplication by a vector field $Z \in \mathcal{T}^{1,0}_M$, as required. □

The following proposition provides a characterization of weak CV-structures which will be useful in the proof of Theorem 28 from the next section. An analogous statement for CDV-structures already appears in the literature (see Theorem 2.1 of [9]).

Proposition 27. *Let $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k})$ be a $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$-structure. Define $\tilde{g}(X, Y) = \tilde{h}(X, \tilde{k}Y)$ as usual and let $\mathcal{E}$ be an eventual identity of the underlying F-manifold $(M, \circ, e)$. Then $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k}, \tilde{\mathcal{U}} = \tilde{C}_{\mathcal{E}})$ extends to a weak CV-structure (i.e. there is an endomorphism $\tilde{\mathcal{Q}}$ of $T^{1,0}M$ such that $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k}, \tilde{\mathcal{U}}, \tilde{\mathcal{Q}})$ is a weak CV-structure) if and only if there is $Z \in \mathcal{T}_M$ such that*

$$L_{\mathcal{E}-\bar{\mathcal{E}}}(\tilde{h})(X, Y) = \tilde{h}(X, Y \circ Z) - \tilde{h}(X \circ Z, Y) \tag{78}$$

and

$$L_{\mathcal{E}}(\tilde{g})(X, Y) = -2\tilde{g}(X \circ Y, Z), \tag{79}$$

hold, for any $X, Y \in \mathcal{T}_M^{1,0}$*. Moreover,* Z *is uniquely determined by* (79) *and satisfies*

$$\tilde{Q} = \tilde{D}_{\mathcal{E}} - L_{\mathcal{E}} + \tilde{C}_Z. \tag{80}$$

Proof. We use an argument similar to the one employed in Theorem 4.5 of [6]. Since $\mathcal{E}$ is an eventual identity, Lemma 26 implies that (76) is satisfied and any endomorphism $\tilde{Q}$ such that $(T^{1,0}M, \tilde{h}, \tilde{k}, \tilde{\mathcal{U}}, \tilde{Q})$ is a weak CV-structure must be of the form (80), with $Z \in \mathcal{T}_M^{1,0}$. Recall, from the definition of weak CV-structures, that $\tilde{Q}$ must be Hermitian with respect to $\tilde{h}$ and skew-symmetric with respect to $\tilde{g}$, i.e.

$$\tilde{h}\big(\tilde{Q}(Y), V\big) = \tilde{h}\big(Y, \tilde{Q}(V)\big) \tag{81}$$

and

$$\tilde{g}\big(\tilde{Q}(Y), V\big) + \tilde{g}\big(Y, \tilde{Q}(V)\big) = 0, \tag{82}$$

must hold, for any $Y, V \in \mathcal{T}_M^{1,0}$. Moreover, (73) must be satisfied as well.

We first show that (78) and (79) are equivalent with (81) and (82) respectively. Since $\tilde{D}$ is the Chern connection of $(T^{1,0}M, \tilde{h})$, for any $X \in \mathcal{T}_M$ and $Y, V \in \mathcal{T}_M^{1,0}$,

$$\begin{aligned} L_X(\tilde{h})(Y, V) &= X\tilde{h}(Y, V) - \tilde{h}(L_X Y, V) - \tilde{h}(Y, L_{\bar{X}} V) \\ &= \tilde{h}\big((\tilde{D}_X - L_X)(Y), V\big) + \tilde{h}\big(Y, (\tilde{D}_{\bar{X}} - L_{\bar{X}})(V)\big). \end{aligned}$$

On the other hand, since X is holomorphic and $\tilde{D}^{(0,1)} = \bar{\partial}$, $L_{\bar{X}} = \tilde{D}_{\bar{X}}$ on $T^{1,0}M$ and we obtain

$$L_X(\tilde{h})(Y, V) = \tilde{h}\big((\tilde{D}_X - L_X)(Y), V\big), \quad \forall Y, V \in \mathcal{T}_M^{1,0}. \tag{83}$$

Similarly,

$$L_{\bar{X}}(\tilde{h})(Y, V) = \tilde{h}\big(Y, (\tilde{D}_X - L_X)(V)\big), \quad \forall Y, V \in \mathcal{T}_M^{1,0}. \tag{84}$$

Relations (83) and (84) with $X := \mathcal{E}$ imply that (78) is equivalent with (81). A similar argument which uses

$$L_X(\tilde{g})(Y, Z) = \tilde{g}\big((\tilde{D}_X - L_X)(Y), Z\big) + \tilde{g}\big(Y, (\tilde{D}_X - L_X)(Z)\big) \tag{85}$$

shows that (79) is equivalent with (82).

Assume now that there is $Z \in \mathcal{T}_M^{1,0}$ (uniquely determined, since $\tilde{g}$ is non-degenerate) such that both (78) and (79) are satisfied and define an endomorphism $\tilde{Q}$ of $T^{1,0}M$ by (80). Then $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k}, \tilde{\mathcal{U}}, \tilde{Q})$ is a weak CV-structure provided that relation (73) is satisfied. We now show that (73) is satisfied if and only if Z is holomorphic. For this we make the following computation: for any $X \in \mathcal{T}_M$,

$$\begin{aligned} \tilde{D}_X(\tilde{D}_{\mathcal{E}} - L_{\mathcal{E}}) - [\tilde{C}_X, \tilde{k}\tilde{C}_{\mathcal{E}}\tilde{k}] &= [\tilde{D}_X, \tilde{D}_{\mathcal{E}} - L_{\mathcal{E}}] + [\tilde{D}_X, \tilde{D}_{\bar{\mathcal{E}}}] \\ &= \tilde{D}_{[X,\mathcal{E}]} - [\tilde{D}_X, L_{\mathcal{E}-\bar{\mathcal{E}}}] \end{aligned} \tag{86}$$

where in the first equality we used

$$[\tilde{C}_X, \tilde{k}\tilde{C}_{\mathcal{E}}\tilde{k}] = -R^{\tilde{D}}_{X,\bar{\mathcal{E}}} = -[\tilde{D}_X, \tilde{D}_{\bar{\mathcal{E}}}] \tag{87}$$

(from the tt^*-equation (53) and $[X, \bar{\mathcal{E}}] = 0$) and in the second equality we used $[\tilde{D}_X, \tilde{D}_{\mathcal{E}}] = \tilde{D}_{[X,\mathcal{E}]}$ because the curvature of $\tilde{D}$ is of type $(1,1)$ and $\tilde{D}_{\bar{\mathcal{E}}} = L_{\bar{\mathcal{E}}}$ on $\mathcal{T}^{1,0}_M$, because $\mathcal{E} \in \mathcal{T}_M$. On the other hand, using (78) and (79) and taking the Lie derivative of $\tilde{g}(X,Y) = \tilde{h}(X, \tilde{k}Y)$ with respect to $\mathcal{E}$ and $\bar{\mathcal{E}}$, we get

$$L_{\mathcal{E}-\bar{\mathcal{E}}}(\tilde{k}) = \tilde{k}\tilde{C}_Z + \tilde{C}_Z\tilde{k} \tag{88}$$

or, equivalently,

$$L_{\mathcal{E}-\bar{\mathcal{E}}}(Y) = -\tilde{k}L_{\mathcal{E}-\bar{\mathcal{E}}}\big(\tilde{k}(Y)\big) + \tilde{C}_Z Y + \tilde{k}\tilde{C}_Z\tilde{k}(Y), \quad \forall Y \in \mathcal{T}^{1,0}_M. \tag{89}$$

From (89), relation (86) becomes

$$\begin{aligned}
\tilde{D}_X(\tilde{D}_{\mathcal{E}} - L_{\mathcal{E}}) - [\tilde{C}_X, \tilde{k}\tilde{C}_{\mathcal{E}}\tilde{k}] &= \tilde{D}_{[X,\mathcal{E}]} + [\tilde{k}\tilde{D}_{\bar{X}}\tilde{k}, \tilde{k}L_{\mathcal{E}-\bar{\mathcal{E}}}\tilde{k}] - [\tilde{D}_X, \tilde{C}_Z + \tilde{k}\tilde{C}_Z\tilde{k}] \\
&= \tilde{D}_{[X,\mathcal{E}]} + \tilde{k}[\tilde{D}_{\bar{X}}, L_{\mathcal{E}-\bar{\mathcal{E}}}]\tilde{k} - \tilde{D}_X(\tilde{C}_Z) - \tilde{k}\tilde{D}_{\bar{X}}(\tilde{C}_Z)\tilde{k} \\
&= \tilde{D}_{[X,\mathcal{E}]} - \tilde{k}\tilde{D}_{\overline{[X,\mathcal{E}]}}\tilde{k} - \tilde{D}_X(\tilde{C}_Z) - \tilde{k}\tilde{D}_{\bar{X}}(\tilde{C}_Z)\tilde{k} \\
&= -\tilde{D}_X(\tilde{C}_Z) - \tilde{k}\tilde{D}_{\bar{X}}(\tilde{C}_Z)\tilde{k}
\end{aligned}$$

where in the first equality we used

$$\tilde{D}_X Y = \tilde{k}\tilde{D}_{\bar{X}}(\tilde{k}Y), \quad \forall X, Y \in \mathcal{T}^{1,0}_M, \tag{90}$$

because $\tilde{k}$ is parallel with respect to the Chern connection $\tilde{D}$; in the third equality we used

$$[\tilde{D}_{\bar{X}}, L_{\mathcal{E}}] = [L_{\bar{X}}, L_{\mathcal{E}}] = L_{[\bar{X},\mathcal{E}]} = 0$$

(because X and $\mathcal{E}$ are holomorphic) and

$$[\tilde{D}_{\bar{X}}, L_{\bar{\mathcal{E}}}] = [\tilde{D}_{\bar{X}}, \tilde{D}_{\bar{\mathcal{E}}}] = \tilde{D}_{\overline{[X,\mathcal{E}]}},$$

because the curvature of $\tilde{D}$ is of type $(1,1)$; in the last equality we used again (90), with X replaced by $[X,\mathcal{E}]$. We deduce that

$$\tilde{D}_X(\tilde{Q}) - [\tilde{C}_X, \tilde{k}\tilde{C}_{\mathcal{E}}\tilde{k}] = -\tilde{k}\tilde{D}_{\bar{X}}(\tilde{C}_Z)\tilde{k}.$$

Therefore, (73) is satisfied if and only if $\tilde{D}_{\bar{X}}(\tilde{C}_Z) = 0$, for any $X \in \mathcal{T}^{1,0}_M$, i.e. Z is holomorphic. Our claim follows. $\square$

6.2.2. Weak CV-structures and duality

Our aim in this section is to prove the following result.

Theorem 28. *Let $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k})$ be a $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$-structure, $\mathcal{E}$ an eventual identity on the underlying F-manifold $(M, \circ, e)$ and $\tilde{\mathcal{U}} := \tilde{C}_{\mathcal{E}}$. Assume that conditions* (60) *and* (62) *are satisfied and let $(T^{1,0}M, C, h, k)$ be the dual $DChk$-structure, as in Theorem 22. Then $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k}, \tilde{\mathcal{U}})$ extends to a weak CV-structure if and only if $(T^{1,0}M, C, h, k, \mathcal{U} := C_e)$ extends to a weak CV-structure.*

Proof. Assume that $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k}, \tilde{\mathcal{U}})$ extends to a weak CV-structure. In order to show that $(T^{1,0}M, C, h, k, \mathcal{U})$ extends to a weak CV-structure, we apply Proposition 27. For this, we need to determine an holomorphic vector field Z such that both (78) and (79) hold, with $\circ$ replaced by the dual multiplication $*$, $\tilde{h}$ replaced by h and $\tilde{g}$ replaced by g (see Section 6.1 for the definitions of $*$, h and g). Define

$$Z := -(\tilde{D}_e e) \circ \mathcal{E} + \frac{1}{2} L_e(\mathcal{E}) \tag{91}$$

and notice that it is holomorphic: from the tt^*-equation (53) and $\tilde{D}^{(0,1)} = \bar{\partial}$, we get, for any $X \in \mathcal{T}_M$,

$$\bar{\partial}_{\bar{X}}(\tilde{D}_e e) = \tilde{D}_{\bar{X}} \tilde{D}_e e = R^{\tilde{D}}_{\bar{X},e} e = [\tilde{C}_e, \tilde{k}\tilde{C}_X \tilde{k}] = 0,$$

because e and X are holomorphic (thus $\tilde{D}_{\bar{X}} e = \bar{\partial}_{\bar{X}} e = 0$ and $[\bar{X}, e] = 0$) and $\tilde{C}_e$ is the identity endomorphism. Therefore, $\tilde{D}_e e$ and hence also Z is holomorphic. We now prove that relations

$$L_e(g)(X, Y) = -2g(X * Z, Y), \quad \forall X, Y \in T^{1,0}M \tag{92}$$

and

$$L_{e-\bar{e}}(h)(X, Y) = h(X, Y * Z) - h(X * Z, Y), \quad \forall X, Y \in T^{1,0}M \tag{93}$$

hold. Taking the Lie derivative with respect to e of the relation

$$g(X, Y) = \tilde{g}\big(X \circ \mathcal{E}^{-1}, Y\big)$$

and using (85) with $X := e$, together with $L_e(\circ) = 0$ and

$$(\tilde{D}_e - L_e)(X) = (\tilde{D}_e e) \circ X, \quad \forall X \in T^{1,0}M \tag{94}$$

(relation (94) is a consequence of the tt^*-equation $\partial^{\tilde{D}} \tilde{C} = 0$, for details see Theorem 4.5 of [6]), we get, for any $X, Y \in \mathcal{T}^{1,0}_M$

$$
\begin{aligned}
L_e(g)(X,Y) &= L_e(\tilde{g})\big(X\circ\mathcal{E}^{-1},Y\big)+\tilde{g}\big(X\circ L_e\big(\mathcal{E}^{-1}\big),Y\big)\\
&= 2\tilde{g}\big((\tilde{D}_e e)\circ\mathcal{E}^{-1}\circ X,Y\big)+\tilde{g}\big(L_e\big(\mathcal{E}^{-1}\big)\circ X,Y\big)\\
&= 2g\big((\tilde{D}_e e)\circ X,Y\big)+g\big(\mathcal{E}\circ L_e\big(\mathcal{E}^{-1}\big)\circ X,Y\big)\\
&= -2g(X*Z,Y),
\end{aligned}
$$

from the definition of $*$. Relation (92) follows. A similar computation shows that (93) holds as well. From Proposition 27, $(T^{1,0}M, C, h, k, \mathcal{U})$ extends to a weak CV-structure, as required. □

6.3. The semi-simple case

Recall that an holomorphic F-manifold $(M,\circ,e)$ is called semi-simple if there are local holomorphic coordinates $(u^1,\dots,u^n)$ on M such that the multiplication $\circ$ is diagonal (see Remark 7 iii)). In the restricted case where the Hermitian metric $\tilde{h}$ and real structure $\tilde{k}$ are also diagonal (and note that in general they need not be so) the various conditions of Theorem 22 are automatically satisfied. More precisely, we can state.

Example 29. Any eventual identity on a semi-simple F-manifold $(M,\circ,e,\tilde{h},\tilde{k})$ with Hermitian metric and real structure taking the form

$$
\frac{\partial}{\partial u^i}\circ\frac{\partial}{\partial u^j}=\delta_{ij}\frac{\partial}{\partial u^j},\qquad \tilde{h}\left(\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j}\right)=H_{ii}\delta_{ij},\qquad \tilde{k}\left(\frac{\partial}{\partial u^i}\right)=k_i\frac{\partial}{\partial u^i}
$$

(where $|k_i|=1$ and $H_{ii}>0$ for any i) automatically satisfies the conditions (60) and (62).

Proof. Recall from Remark 7 iii) that any eventual identity on $(M,\circ,e)$ is given by $\mathcal{E}=\sum_{i=1}^n f_i\frac{\partial}{\partial u^i}$, where f_i depends on the variable u^i only and is holomorphic and non-vanishing. We will check (60) for fundamental vector fields $X=\frac{\partial}{\partial u^i}$, $Y=\frac{\partial}{\partial u^j}$ $(i\neq j)$ and $Z=\frac{\partial}{\partial u^p}$. Since the multiplication is semi-simple, (60) is clearly satisfied if $p\notin\{i,j\}$. If $p=i$ say, then (60) becomes

$$
\tilde{D}_{\frac{\partial}{\partial u^j}}\left(f_i\frac{\partial}{\partial u^i}\right)=\mathcal{E}\circ\tilde{D}_{\frac{\partial}{\partial u^j}}\left(\frac{\partial}{\partial u^i}\right),
$$

or, since f_i depends only on u^i and $i\neq j$,

$$
f_i\tilde{D}_{\frac{\partial}{\partial u^j}}\left(\frac{\partial}{\partial u^i}\right)=\mathcal{E}\circ\tilde{D}_{\frac{\partial}{\partial u^j}}\left(\frac{\partial}{\partial u^i}\right). \tag{95}
$$

On the other hand, since $\tilde{h}$ is diagonal in $(u^1,\dots,u^n)$, its Chern connection has the form

$$
\tilde{D}_X\left(\frac{\partial}{\partial u^i}\right)=\partial_X\log(H_{ii})\frac{\partial}{\partial u^i},\quad \forall X\in T^{1,0}M,\ \forall i.
$$

In particular, $\tilde{D}_{\frac{\partial}{\partial u^j}}(\frac{\partial}{\partial u^i})$ is a multiple of $\frac{\partial}{\partial u^i}$ and (95) follows. We proved that relation (60) holds. It remains to prove relation (62). From the definitions of the real structure and multiplication, it can be checked that for any $Y := \sum_{i=1}^{n} Y^i \frac{\partial}{\partial u^i}$, the composition $\tilde{k}\tilde{C}_Y\tilde{k}$ is the multiplication by the vector $\sum_{i=1}^{n} \overline{Y^i} \frac{\partial}{\partial u^i}$. In particular, both sides of (62) vanish. Our claim follows. □

It should be pointed out that Eqs. (60) and (62) place highly restrictive conditions on the various structures and may, in general, have no solution (as happens for some of the two-dimensional non-semi-simple examples in [18]). Just as almost-dual Frobenius manifolds satisfy almost all of the axioms of a Frobenius manifold, asking for the twisted structures to satisfy the full tt^* axioms may be too restrictive a condition. However, the above example does show that solutions in the semi-simple case – albeit in the subclass of diagonal real and Hermitian structures – do exist.

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